Zigzag structure in triangulations of surfaces

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I hereby declare that the dissertation is my own work.

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The dissertation is ready to be reviewed.

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Abstract

In this thesis, we investigate zigzags in triangulations of surfaces. We introduce the concept of z-monodromy and show that there are precisely 7 types (M1)–(M7) of z-monodromies of faces in triangulations. We provide examples for all these types.

A triangulation is z-knotted (i.e. it contains a single zigzag up to reversing) if and only if the z-monodromy of each face is of one of the type (M1)–(M4). Using this fact we show that each triangulation admits a z-knotted shredding. The proof is constructive.

Another result related to z-monodromies which we prove states that the z-monodromies (M1) and (M2) are exceptional. For each $i \geq 3$ there is a triangulation with the z-monodromy of type (Mi) for all faces. For (M1) and (M2) this fails: all faces with the z-monodromy of one of these types form a forest in the dual graph.

We investigate z-oriented triangulations, i.e. triangulations with a direction chosen on each zigzag. There are precisely two types of faces in such triangulations. We show that each z-oriented triangulation admits a z-oriented shredding with all faces of the first type. We will focus only on such triangulations. An important subclass is formed by so called z-homogeneous triangulations. We describe a one-to-one correspondence between z-homogeneous triangulations and embeddings of Eulerian digraphs in surfaces. We show that a z-oriented triangulation (with all faces of the first type) provide a decomposition of the surface into connected components of the following three types: open discs, open cylinders and open Möbius strips. The triangulation is z-homogeneous if and only if all connected components are open discs.

Since z-knotted triangulations have a single z-orientation (up to reversing), we can say on z-homogeneity of z-knotted triangulations without fixing a z-orientation. We propose an algorithm of constructing of such a triangulation from an arbitrary z-homogeneous triangulation. This construction is based on the z-monodromies of pairs of edges.

Keywords: embedded graph, triangulation, zigzag, z-monodromy, z-knotted triangulation, z-orientation, z-homogeneous triangulation.

Streszczenie

W niniejszej rozprawie doktorskiej badam zygzaki w triangulacjach powierzchni. Wprowadzam pojęcie z-monodromii i pokazuję, że istnieje dokładnie 7 typów z-monodromii ścian w triangulacjach. Podaję przykłady dla każdego z tych typów.

Triangulacja jest z-zawiązana (tzn. zawiera dokładnie jeden zygzak z dokładnością do odwrotności) wtedy i tylko wtedy, gdy z-monodromia każdej ściany jest jednego z typów (M1)–(M4). Wykorzystując ten fakt wykazuję, że każda triangulacja ma z-zawiązane rozdrobnienie. Dowód tego twierdzenia jest konstruktywny.

Inny wynik związany z z-monodromiami który dowodzę, stwierdza, że z-monodromie (M1) i (M2) są wyjątkowe. Dla każdego $i \geq 3$ istnieje triangulacja z z-monodromiami typu (Mi) dla każdej ściany. Dla (M1) oraz (M2) nie jest to prawdą: wszystkie ściany z z-monodromią jednego z tych dwóch typów tworzą las w dualnym grafie.

Badam z-zorientowane triangulacje, tzn. triangulacje z kierunkiem wybranym na każdym z zygzaków. Istnieją dokładnie dwa typy ścian w takich triangulacjach. Wykazuję, że każda z-zorientowana triangulacja ma z-zorientowane rozdrobnienie, w którym wszystkie ściany są pierwszego typu. Będę się koncentrował tylko na takich triangulacjach. Ważną podklasę stanowią tzw. z-jednorodne triangulacje. Opisuję wzajemnie jednoznaczną odpowiedniość pomiędzy z-jednorodnymi triangulacjami i zanurzeniami digrafów eulerowskich w powierzchnie. Pokazuję, że z-zorientowana triangulacja (ze wszystkimi ścianami typu pierwszego) wyznacza rozkład powierzchni na składowe spójności trzech typów: otwarte dyski, otwarte cylindry oraz otwarte wstęgi Möbiusa. Triangulacja jest z-jednorodna wtedy i tylko wtedy, gdy wszystkie składowe spójności są otwartymi dyskami.

Ponieważ z-zawiązane triangulacje posiadają dokładnie jedną z-orientację (z dokładnością do odwrotności), możemy mówić o z-jednorodności bez przywiązania do z-orientacji. Proponuję algorytm konstruowania takiej triangulacji z dowolnej z-jednorodnej triangulacji. Ta konstrukcja opiera się na z-monodromiach par krawędzi.

Słowa kluczowe: zanurzenie grafu, triangulacja, zygzak, z-monodromia, z-zawiązana triangulacja, z-orientacja, z-jednorodna triangulacja.

1 Introduction

In this thesis, we investigate *zigzags* in graphs embedded in surfaces, in particular, zigzags in triangulations. Zigzags generalize the classical concept of *Petrie polygons* in regular polyhedra. A Petrie polygon is a skew polygon formed by sides of a regular polyhedron such that two consecutive sides, but no three, belong to a face [5, p. 24]. It was a useful tool to study regular polyhedra (see [5] for the details).

In a similar way, we can define sequences of edges in graphs embedded in surfaces. They are called *zigzags* [7, 17] or *closed left-right paths* [11, 29]. In contrast to regular polyhedra, zigzags of embedded graphs can be self-intersected in vertices and edges. Various kinds of results concerning zigzags in *planar graphs*, i.e. graphs embedded in a sphere, are presented in [7] and [11, Chapter 17]. The case when a graph is embedded in an arbitrary surface is not well-studied.

Analogs of zigzags in objects of dimension greater than 3 are investigated in [5, 7, 8, 32].

Zigzags are important for many reasons. Now, we provide some of them.

(1). Zigzags are used in computer graphics [13] and mathematical chemistry [7, Chapter 2]. For example, zigzags were successfully exploited to enumerating all combinatorial possibilities for fullerenes [3] (fullerenes are considered as spherical embeddings of 3-regular graphs whose faces are 5- and 6-gons).

(2). *Z-knotted embedded graphs*, i.e. embedded graphs containing a single zigzag (up to reversing), are closely connected to the *Gauss code problem.* Recall that a *Gauss code* is a word (a sequence of symbols) where each symbol occurs precisely twice. A closed curve with simple self-intersections embedded in a closed 2-dimensional surface can be identified with the Gauss code whose symbols are the intersection points. The problem is to characterize all Gauss codes that realize as curves. In the spherical case the solution is well-known, see [10, 19, 20, 26, 27, 28] and [11, Section 17.7]. For other surfaces the problem is partially solved [6, 18]. The z-knottedness of an embedded graph Γ is equivalent to an existence of a zigzag which passes through each edge precisely twice, i.e. this zigzag is a Gauss code whose symbols are edges of Γ. The *medial graph* $\mathrm{M}(\Gamma)$ is the graph (embedded in the same surface) whose vertices are edges of Γ and two edges are adjacent in $\mathrm{M}(\Gamma)$ if they have a common vertex and belong to the same face of Γ. The unique zigzag of Γ corresponds to a closed walk in $\mathrm{M}(\Gamma)$ which is a curve representing a certain Gauss code.

(3). The classical duality interchanges vertices and faces of graphs embedded in surfaces and preserves edges (zigzag also are preserved). Using vertices, faces and zigzags, Lins [17] described another two dualities and two trialities (some of them were previously found by Wilson [34]). For example, *Petrie duality* preserves vertices and interchanges zigzags and faces. It was proved in [14] that there is no other "good" notion of a duality/triality (for graphs embedded in surfaces) than the ones described in [17]. In [12], zigzags together with Petrie duality were used to distinguishing all 14 types of locally finite, planar, 3-connected edge-transitive graphs.

(4). A cycle (in a graph embedded in a surface) is called a *bicycle* if edges of this cycle form a cycle in the dual graph. All bicycles generate a vector space over $\mathbb{Z}_2$. In a planar graph the dimension of this vector space is equal to $n-1$, where n is the number of zigzags (up to reversing); in particular, a planar graph is z-knotted if and only if it does not contain non-trivial bicycles [29], see also [11, Theorem 17.3.5]. In addition, Shank [29] obtained the following characterization of z-knottedness: a planar graph is z-knotted if and ony if its number of spanning trees is odd. For non-planar case the dimension of the vector space formed by bicycles depends on the number of zigzags and the Euler characteristic [6].

It was noted above that zigzags are detailed investigated in [7, 11]. A large portion of results concerns zigzags in planar graphs. We investigate zigzags in triangulations of surfaces (not necessarily orientable). These objects are dual to 3-regular graphs considered in [7]. Since the duality preserves all zigzags, our research continues [7]. We prefer triangulations instead of 3-regular graphs for some technical reasons. Other our reason is that zigzags of triangulations are used in computer graphics.

The thesis is organised as follows.

In Section 2, we describe briefly all basic concepts: graphs embedded in surfaces, triangulations, zigzags and z-orientations.

In Section 3, we consider the *z-monodromy of a face* (a map of the first return of a zigzag to a face). We present a classification of z-monodromies and provide examples showing that all possibilities from this classification are realized. By the classification, there are precisely 7 types of z-monodromies (Theorem 2). Four of these types occur in z-knotted triangulations. Conversely, a triangulation is z-knotted if the z-monodromy of each face is of one of the four types (Theorem 3).

In Section 4, we show that every triangulation admits a z-knotted shredding (Theorem 4). A large class of examples of z-knotted 3-regular planar graphs (equivalently, z-knotted triangulations of a sphere) were obtained using computer [7]. The main result of this section gives a purely mathematical construction of z-knotted triangulations of an arbitrary surface. The concept of z-monodromy plays an important role in this construction. As another application of the z-monodromy, we present a shorter proof of the main result of [23] which describes all possibilities when the connected sum of two z-knotted triangulations is z-knotted (Theorem 5).

In Section 5, we investigate *z-oriented triangulations* where directions of all zigzags are distinguished. In such a triangulation there are two types of edges: *type I* (zigzags pass through an edge in two different directions) and *type II* (zigzags pass through an edge in the same direction). It is not difficult to prove that there are two types of faces: *type I* (precisely two edges are of type I and the remaining edge is of type II) and *type II* (all edges are of type II). We restrict ourself to the case when all faces are of type I for the reason that any z-oriented triangulation admits such a shredding. We are especially interested in *z-homogeneous triangulations* (in each zigzag after every edge of type II there are precisely two edges of type I). These triangulations are closely connected to embeddings of *Eulerian digraphs* (see [2, 9, 21] for embeddings in a sphere and [1, 4] for embeddings in an arbitrary surface). There is a one-to-one correspondence between z-homogeneous triangulations and embeddings of Eulerian digraphs (Theorem 6). We remove all edges of type II from a surface. In the general case, after this operation the surface is decomposed in open discs, open cylinders and open Möbius strips (Theorem 7). Our triangulation is z-homogeneous if and only if the surface is decomposed in open discs only (Theorem 6).

The construction described above depends on a z-orientation. On the other hand, z-knotted triangulations have precisely one z-orientation up to reversing (the reversing of z-orientation does not change the types of edges and faces). For this reason, we can say on z-homogeneity of z-knotted triangulations without fixing a z-orientation. In Section 6, we describe an algorithm which produces a z-knotted and z-homogeneous triangulation from an arbitrary z-homogeneous triangulation (Theorem 8). Our construction will be based on a modification of the concept of z-monodromy to a pair of edges. Such z-monodromies are more complicated than the z-monodromies of faces: there are precisely 13 classes of z-monodromies.

In Section 7, we return to z-monodromies of faces. We mentioned that there are 7 types of z-monodromies of faces (M1)–(M7). For every $i = 3, \dots, 7$ there is a triangulation where the z-monodromy of each face is of type (Mi). The types (M1) and (M2) are exceptional: in the dual graph all faces with one of these types of z-monodromies form a forest (Theorem 9). Consequently, for $i = 1, 2$ there exist no triangulation where the z-monodromies of all faces are of type (Mi).

In conclusion, the main contributions of the thesis are the following:

- the concept of z-monodromy;
- every triangulation admits a z-knotted shredding;
- z-homogeneous triangulations related to embeddings of Eulerian digraphs;
- an algorithm of constructing of z-knotted and z-homogeneous triangulations on an arbitrary surface.

All author's results presented in this thesis were obtained in the following papers: [24, 25, 30, 31].

2 Basic concepts and constructions

2.1 Surfaces and connected sums

A *closed surface* (or simply a *surface*) is a connected compact Hausdorff topological space which is locally homeomorphic to an open 2-dimensional disc. Each surface can be constructed in the following way. We take a *fundamental polygon*, i.e. a polygon with even number of oriented sides. Next, we split the set of sides into pairs and identify sides from the same pair according to their orientations.

Example 1. Examples of elementary surfaces obtained from fundamental squares are presented below.

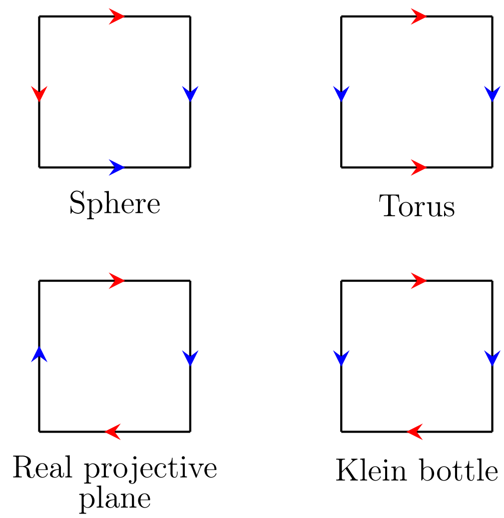

Figure 1

Following [22], we introduce the concept of *graphs embedded into surfaces*. Let us take a finite family of pairwise disjoint polygons such that all of them together have an even number of sides. Each side can be identified with a pair of its endpoints. We order each of these pairs of endpoints in an arbitrary way and choose any partition of the family of all sides into pairs. We identify each two edges from the same pair according to their orientations and obtain a certain topological space S. If S is connected, then it is a surface. In this case, the set of (non-oriented) sides and the set of their endpoints can be considered as the set of *edges* and the set of *vertices* of a connected multigraph Γ contained in S. This construction is known as an *embedding of* Γ *in* S. The family of polygons is called the set of *faces* of Γ. The interior of each face is homeomorphic to an open 2-dimensional disc. In the case when each face is homeomorphic to a closed 2-dimensional disc, we say that the embedding is a *closed 2-cell embedding.* If S is obtained from family of triangles and Γ is a simple graph, then Γ is said to be a *triangulation* of S. Any surface admits a triangulation (see [22]).

Let Γ be a graph embedded in a surface S. The *dual graph* Γ^* is a graph whose vertices are faces of Γ and whose edges are formed by pairs of distinct faces whose intersection contains an edge from Γ. Such pairs of faces will be called *adjacent.* The graph Γ^* is embeddable in S in the following way: we take a point in the interior of each face, two such points are connected by a simple curve if and only if the corresponding faces are adjacent. The duality provides a one-to-one correspondence between the faces of Γ and the vertices of Γ^* and a one-to-one correspondence between the vertices of Γ and the faces of Γ^*. Note that $\Gamma^{**} = \Gamma$. In particular, triangulations are dual to cubic graphs and vice versa. A special class of embedded cubic graphs is formed by fullerenes, i.e. embeddings of 3-regular simple graphs in a sphere whose faces are pentagons and hexagons, see [7]. Thus, fullerenes are dual to triangulations of a sphere whose vertex degrees are 5 or 6.

Example 2. The basic examples of graphs embedded in a sphere are graphs of Platonic solids, see Fig. 2. We see that a tetrahedron, a octahedron and a icosahedron are triangulations. A tetrahedron, a cube and a dodecahedron are cubic graphs and a dodecahedron is a fullerene. A cube and a icosahedron are dual to a octahedron and a dodecahedron, respectively. A tetrahedron is self-dual.

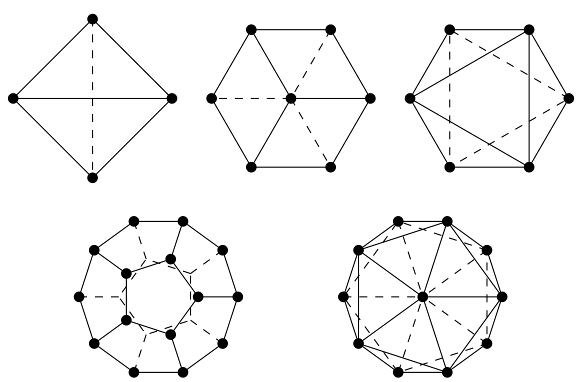

Figure 2

Consider the topological space obtained from a square by gluing two opposite sides together, see Fig. 3. This topological space is called a *Möbius strip.* This is an example of a *surface with boundary*, i.e. a Hausdorff topological space whose each point has a neighbourhood homeomorphic to some open subset of a closed half-plane.

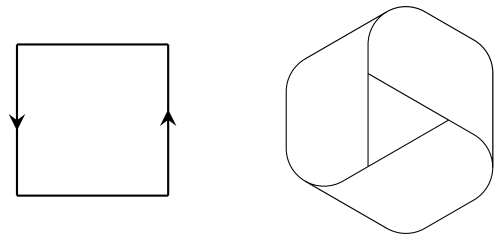

Figure 3

A closed surface is said to be *non-orientable* if it contains a subset homeomorphic to a Möbius strip; otherwise, we call it an *orientable* surface (see, for example, [15]).

Now, we describe a method of constructing new surfaces from a pair of surfaces. Let S and S' be surfaces. We remove open discs D, D' from each of them (respectively). We take any homeomorphism $g : \partial D \to \partial D'$ and identify boundaries according to it. As a result, we get a new surface denoted by $S \# S'$ and called the *connected sum of surfaces* S and S' (for another homeomorphism $g' : \partial D \to \partial D'$ we obtain a homeomorphic surface). A sphere $\mathbb{S}$ is an identity element of the connected sum operation, i.e. $S \# \mathbb{S} \simeq S$ for any surface S.

Example 3. Consider a torus $\mathbb{T}$. The connected sum of two tori is called a *double torus* and denoted by $\mathbb{T}_2$ (see Fig. 4).

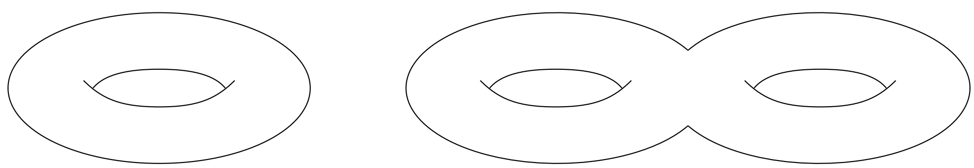

Figure 4

Similarly, the surface obtained by consecutive connected sums of n tori is called an n*-torus* and denoted by $\mathbb{T}_n$ (where $\mathbb{T}_1 = \mathbb{T}$). This is orientable surface for any n. Consider examples of surfaces that are non-orientable. Let $\mathbb{P}$ be a real projective plane. Then the connected sum $\mathbb{P} \# \mathbb{P}$ is a Klein bottle $\mathbb{K}$. As above, we can glue $2n$ copies of $\mathbb{P}$ and obtain the surface denoted by $\mathbb{K}_n$ (where $\mathbb{K}_1 = \mathbb{K}$) which is also homeomorphic to the connected sum of n copies of $\mathbb{K}$. The connected sum $\mathbb{K} \# \mathbb{P}$ is homeomorphic to the connected sum $\mathbb{T} \# \mathbb{P}$, see [15].

The following theorem gives a classification of closed surfaces (see, for example, [15] for more details).

Theorem 1. *Every closed surface is homeomorphic to one of the following:*

(1) *a sphere,*

(2) *the connected sum of tori,*

(3) *the connected sum of real projective planes.*

Let Γ and Γ' be triangulations of surfaces S and S' (respectively) and D and D' be faces of these triangulations. Let $g : \partial D \to \partial D'$ be a homeomorphism which transfers each vertex to a vertex. Such homeomorphisms will be called *special*. The gluing by this homeomorphism gives a triangulation of $S \# S'$ called the *connected sum of triangulations* Γ and Γ' which will be denoted by $\Gamma \#_g \Gamma'$. In contrast to the connected sum of surfaces, the result depends on a choice of homeomorphism.

Example 4. The *n-bipyramid BP_n* is a triangulation of a sphere consisting of an n-cycle (called the *base* of the n-bipyramid) whose vertices are denoted by $1, \dots, n$, and the remaining two vertices a, b connected by edges with all vertices of the base. The bipyramid BP_3 is the connected sum of two tetrahedra for every special homeomorphism, see Fig. 5.

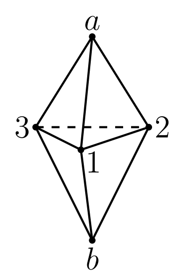

Figure 5

On the other hand, the connected sum of two n-bipyramids is one of three different triangulations of a sphere (see Fig. 6 for the case $n = 3$). Note that the last two triangulations in Fig. 6 are embeddings of the same graph, but in different ways.

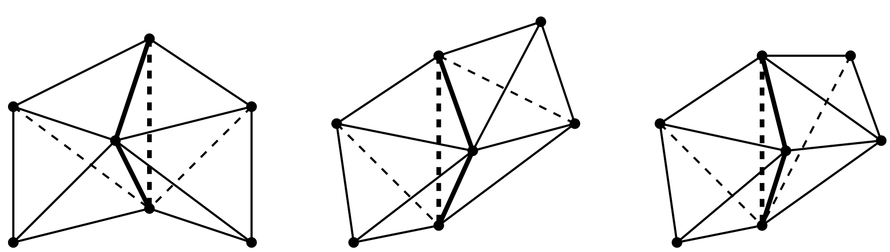

Figure 6

2.2 Zigzags

Let Γ be a graph embedded in a surface S. We will always require that the following assertions are fulfilled:

(1) every edge is contained in precisely two distinct faces,

(2) the intersection of two distinct faces is an edge or a vertex or the empty set.

Remark 1. The both conditions are fulfilled when Γ is a triangulation. The fulfillment of the first condition is obvious. If the second condition fails for a triangulation, then the corresponding embedded graph has a double-edge which contradicts the fact that the triangulation is a simple graph embedding.

Recall that two distinct faces are adjacent if their intersection is an edge. If two distinct edges have a common vertex and there is a face containing them, the edges are called *adjacent*. A *zigzag* in Γ is a sequence of edges $Z = \{e_i\}_{i\in\mathbb{N}}$ where for every $i \in \mathbb{N}$ the following two conditions hold:

(Z1) e_i, e_{i+1} are adjacent,

(Z2) the faces containing e_i, e_{i+1} and e_{i+1}, e_{i+2} are adjacent.

See Fig. 7.

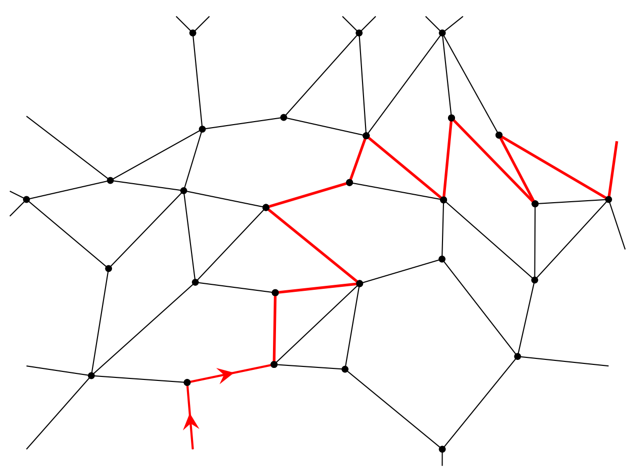

Figure 7

Note that the edges e_i, e_{i+2} are disjoint. Furthermore, e_i, e_{i+1} uniquely determine e_{i+2}. So, the zigzag Z is completely determined by the pair e_i, e_{i+1} for every $i \in \mathbb{N}$, i.e. there is a unique zigzag containing the subsequence e_i, e_{i+1}. This means that each such ordered pair defines one of zigzags in Γ. Since Γ has a finite number of edges and faces, there is a natural number $n > 0$ such that

$$e_{n+1} = e_1 \text{ and } e_{n+2} = e_2.$$

Then we have

$$e_{i+n} = e_i \text{ for all } i \in \mathbb{N}.$$

The smallest such number n is the *length* of Z. Thus our zigzag Z can be considered as the *cyclic sequence* $e_1, e_2, \ldots, e_n$.

Example 5. Figure 8 shows zigzags in Platonic solids (marked with a bold line).

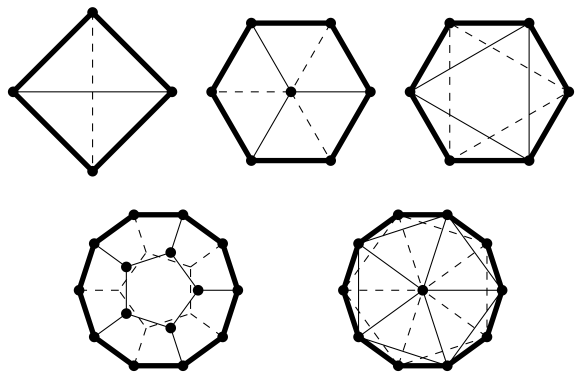

Figure 8

The zigzag $Z = \{e_1, e_2, \ldots, e_n\}$ can be written as the cyclic sequence of faces $F_1, F_2, \ldots, F_n$, where F_i is the face containing e_i, e_{i+1} or as the cyclic sequence of vertices $v_1, v_2, \ldots, v_n$, where v_i is the common vertex of e_i and e_{i+1} (for every $i \in \{1, 2, \ldots, n\}$ and $e_{n+1} = e_1$). Thus, each zigzag from Γ can be considered as a zigzag in Γ^* and vice versa.

If all edges in the cyclic form of Z are distinct, then Z is said to be *edge-simple*. Similarly, if all vertices in the cyclic form of Z are distinct, then Z is *vertex-simple*. If Z is vertex-simple, then it is also edge-simple, but the converse is not true. For example, all zigzags in each of Platonic solids are vertex-simple, see Example 5. Triangulations with zigzags which are edge-simple but not vertex-simple will be given in Example 6.

Let $X = \{e_1, \ldots, e_n\}$ be a sequence of edges. We denote the *reversed* sequence $\{e_n, \ldots, e_1\}$ by X^{-1}. In the case when X is a zigzag, the reversed sequence X^{-1} also is a zigzag.

Proposition 1. *$Z \neq Z^{-1}$ for every zigzag Z.*

Proof. Suppose that Z contains a sequence e, e'. Then Z^{-1} contains e', e. If $Z = Z^{-1}$, then this zigzag contains the sequence $e, e', \ldots, e', e$. Therefore, Z is either the sequence

$$\ldots, e, e', e_1, e_2, \ldots, e_m, e_m, \ldots, e_2, e_1, e', e, \ldots$$

or the sequence

$$\ldots, e, e', e_1, e_2, \ldots, e_{m-1}, e_m, e_{m-1}, \ldots, e_2, e_1, e', e, \ldots$$

Each of these cases is impossible by the definition of zigzag. $\square$

Example 6. We describe zigzags in bipyramids.

(a). Suppose that $n = 2k + 1$. Then the bipyramid BP_n has a single zigzag (up to reversing). If k is odd, then the zigzag is

$$a1, 12, 2b, b3, \ldots, a(n-2), (n-2)(n-1), (n-1)b, bn, n1,$$
$$1a, a2, 23, 3b, \ldots, a(n-1), (n-1)n, nb,$$
$$b1, 12, 2a, a3, \ldots, b(n-2), (n-2)(n-1), (n-1)a, an, n1,$$
$$1b, b2, 23, 3a, \ldots, b(n-1), (n-1)n, na.$$

If k is even, then this zigzag is

$$a1, 12, 2b, b3, \ldots, b(n-2), (n-2)(n-1), (n-1)a, an, n1,$$
$$1b, b2, 23, 3a, \ldots, a(n-1), (n-1)n, nb,$$
$$b1, 12, 2a, a3, \ldots, a(n-2), (n-2)(n-1), (n-1)b, bn, n1,$$
$$1a, a2, 23, 3b, \ldots, b(n-1), (n-1)n, na.$$

It is easy to calculate that the length of each of these zigzags is twice the number of edges of the bipyramid.

(b). For $n = 2k$, where k is odd, there are precisely two zigzags (up to reversing):

$$a1, 12, 2b, b3, 34, \ldots, a(n-1), (n-1)n, nb,$$
$$b1, 12, 2a, a3, 34, \ldots, b(n-1), (n-1)n, na$$

and

$$a2, 23, 3b, b4, 45, \ldots, an, n1, 1b,$$
$$b2, 23, 3a, a4, 45, \ldots, bn, n1, 1a.$$

(c). If $n = 2k$ and k is even, then the bipyramid contains precisely four zigzags (up to reversing):

$$a1, 12, 2b, \ldots, b(n-1), (n-1)n, na;$$
$$b1, 12, 2a, \ldots, a(n-1), (n-1)n, nb;$$
$$a2, 23, 3b, \ldots, bn, n1, 1a;$$
$$b2, 23, 3a, \ldots, an, n1, 1b.$$

In the case (c) all zigzags are edge-simple. For the remaining cases this fails. Note that all zigzags of 4-bipyramid (the octahedron) are vertex-simple, but $(2k)$-bipyramids for $k = 4, 6, 8, \ldots$ are not vertex-simple.

2.3 Z-oriented triangulations

The concept of z-orientation for embedded graphs is described in [7]. We restrict ourself to the case of triangulations.

Let Γ be a triangulation of a surface S. By Proposition 1, the triangulation Γ contains even number of zigzags, lets say $2k$ (or k zigzags up to reversing). A *z-orientation* of Γ is a collection τ of k zigzags which does not contain any pair of mutually reversed zigzags. In other words, $Z \in \tau$ or $Z^{-1} \in \tau$ for each zigzag Z. There are 2^k z-orientations for Γ. If $\tau = \{Z_1, \dots, Z_k\}$, then the z-orientation $\tau^{-1} = \{Z_1^{-1}, \dots, Z_k^{-1}\}$ is *reversed* to τ. A pair (Γ, τ) consisting of the triangulation and one of its z-orientations is called a *z-oriented triangulation*.

Our triangulation will be called *z-knotted* if $k = 1$, i.e. it has a single zigzag up to reversing. Such triangulations have precisely two mutually reversed z-orientations.

Proposition 2. *If τ is a z-orientation of Γ, then for every edge e one of the following possibilities is realized:*

(1) *there is a zigzag $Z \in \tau$ such that e occurs in Z twice and the remaining zigzags from τ do not contain e;*

(2) *there are two distinct zigzags $Z, Z' \in \tau$ such that e occurs in each of them only once and the remaining zigzags from τ do not contain e.*

Proof. There are two distinct faces containing e. Suppose that the remaining edges of the first face are e_1, e_2 and of the second face are e'_1, e'_2. Without loss of generality, we may assume that zigzags of Γ passes through e in the four following ways

$$\begin{array}{ccc} \dots, e_1, e, e'_2, \dots, & \qquad & \dots, e'_1, e, e_2, \dots, \\ \dots, e'_2, e, e_1, \dots, & \qquad & \dots, e_2, e, e'_1, \dots \end{array}$$

and we see that zigzags in the same column are reversed (see Fig. 9).

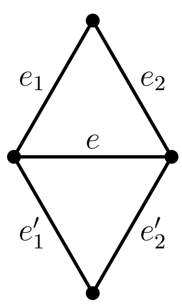

Figure 9

Thus, zigzags from τ contain only one sequence from each of the above columns. Denote these zigzags by Z and Z' for the first and the second column respectively. The first case from the proposition is satisfied for $Z = Z'$. Otherwise, we get the second case. □

Corollary 1. *If τ is a z-orientation of Γ, then the set of edges is double covered by the zigzags of τ, i.e. every edge is contained in two distinct zigzgas from τ or it occurs in one zigzag of τ twice. In particular, Γ is z-knotted if and only if it contains a zigzag passing through every edge twice.*

Corollary 2. *For any z-orientation τ, the sum of lengths of all zigzags from τ is equal to the twice number of edges in Γ.*

Now, we fix a certain z-orientation τ. We say that e is an *edge of type I* or an *edge of type II* if zigzags from τ passes through e twice in different directions or twice in the same direction, respectively. We will consider edges of type II together with the direction induced by τ. The subgraph of Γ consisting of all edges of type II and their vertices will be denoted by Γ_{II} and considered as a *directed graph* embedded in S. A vertex of Γ is *of type I* if all edges incident to this vertex are of type I. Otherwise, the vertex is *of type II*. Thus, the vertex set of Γ_{II} is formed by all vertices of type II.

Lemma 1. *For each vertex of type II the number of edges of type II which enter to this vertex is equal to the number of edges of type II which leave it.*

Proof. Let v be any vertex of type II. For each zigzag from τ passing through v, the number of times that the zigzag enters to v is equal to the number of times that this zigzag leaves v. □

Proposition 3. *For each face one of the following possibilities is realized:*

(1) *the face contains two edges of type I and the third edge is of type II, see* Fig. 10 (a)*;*

(2) *all edges of the face are of type II and form a directed cycle, see* Fig. 10 (b).

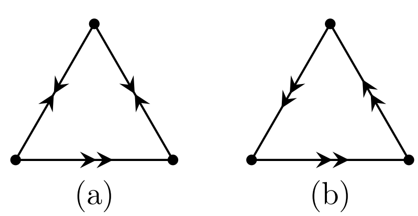

Figure 10

The face is *of type I* in the first case and *of type II* in the second case.

Proof. Consider a face whose edges are e_1, e_2, e_3. We can assume that the zigzag containing the sequence e_1, e_2 belongs to τ. Let Z, Z' be the zigzags containing the sequences e_2, e_3 and e_3, e_1 (respectively). We get four cases depending on whether Z or Z' belong to τ. It is easy to check that each of them corresponds to (1) or (2). $\square$

Observe that the reversing of a z-orientation (each zigzag from the z-orientation is replaced by the reversed) does not change types of edges and, consequently, types of vertices and faces. However, the directions of edges of type II are reversed.

Example 7. We return to the n-bipyramids from Example 6.

(a). Odd-gonal bipyramids are z-knotted and have two mutually reversed z-orientations. For both these z-orientations the edges ai and bi, $i \in \{1, \dots, n\}$, are of type I. The edges of the base are of type II and Γ_{II} is a directed cycle. So, the vertices $1, \dots, n$ are of type II and a, b are of type I. All faces are of type I.

(b). Consider the case $n = 2k$, where k is odd. If the z-orientation consists of two zigzags presented in Example 6 (b), this is a situation similar to the previous case: Γ_{II} is the base, a and b are of type I and all faces are of type I. However, if we replace one of these zigzags by its reversion, then all faces, edges and vertices will be of type II.

(c). Let $n = 2k$ and k be even. Denote the zigzags presented in Example 6 (c) by Z_1, Z_2, Z_3, Z_4 (keeping the order from the example). If the z-orientation consists of these zigzags, then types of faces, edges and vertices are as in (a) and in the first case from (b). For the z-orientation

$$\{Z_1, Z_2, Z_3^{-1}, Z_4^{-1}\}$$

all faces are of type II, but in the z-orientation

$$\{Z_1, Z_2, Z_3, Z_4^{-1}\}$$

the both types of faces occur.

3 Z-monodromy

Let Γ be a triangulation of a surface. Now, we investigate the concept of z-monodromy for faces of Γ. In this section, the triangulation Γ is considered without any z-orientation. All results described in this section come from [25]. The z-monodromy is a crucial tool used to prove the main result of the next section which concerns the existence of a z-knotted shredding for any triangulation. We return to the face z-monodromy in Section 7. In Section 6, we will consider z-monodromies for pairs of edges.

3.1 Definition and basic properties

Let F be a face of Γ whose vertices are a, b, c and let $\Omega(F)$ be the set of all oriented edges of F:

$$\Omega(F) = \{ab, bc, ca, ac, cb, ba\}.$$

We write xy for the edge from $x \in \{a, b, c\}$ to $y \in \{a, b, c\}$. If e is the edge xy, then the edge yx is denoted by $-e$.

Define the permutation D_F on the set $\Omega(F)$ as the following composition of two commuting 3-cycles

$$D_F = (ab, bc, ca)(ac, cb, ba).$$

In other words, if x, y, z are three mutually distinct vertices of F, then $D_F(xy) = yz$. If $D_F(e) = e'$, then we have the equality $D_F(-e') = -e$.

Now, we introduce the notion of *z-monodromy* of the face F. Let $e \in \Omega(F)$ and take $e_0 \in \Omega(F)$ such that $D_F(e_0) = e$. Since each of zigzags is completely determined by any pair of consecutive edges, there is precisely one zigzag Z containing e_0, e. We define $M_F(e)$ as the first element of $\Omega(F)$ contained in Z after the sequence e_0, e (see Fig. 11).

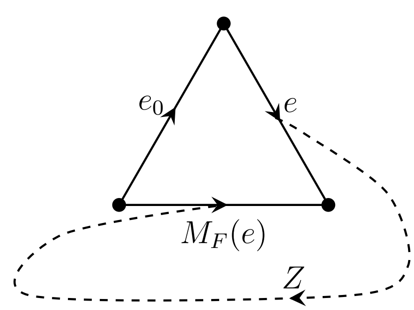

Figure 11

Denote by $\mathcal{Z}(F)$ the set of all zigzags containing the sequence $e, D_F(e)$, where $e \in \Omega(F)$.

Lemma 2. *Let Z be a zigzag. The following assertions are fulfilled:*

(1) *$Z \in \mathcal{Z}(F)$ if and only if Z contains at least one edge of F.*

(2) *If $Z \in \mathcal{Z}(F)$, then $Z^{-1} \in \mathcal{Z}(F)$.*

(3) *$|\mathcal{Z}(F)|$ is equal to 2 or 4 or 6.*

Proof. (1). Assume that an edge of F consists of vertices $x, y \in \{a, b, c\}$ and Z passes through this edge from x to y. Then either Z contains the sequence $xy, D_F(xy)$ or Z contains the sequence e, xy, where $D_F(e) = xy$. In both cases $Z \in \mathcal{Z}(F)$.

(2). Let Z belongs to $\mathcal{Z}(F)$. Then Z contains the sequence e, e', where $e \in \Omega(F)$, $e' = D_F(e)$ and Z^{-1} contains the sequence $-e', -e$. Since $D_F(-e') = -e$, the zigzag Z^{-1} belongs to $\mathcal{Z}(F)$.

(3). The set $\Omega(F)$ consists of 6 elements, but for some distinct $e, e' \in \Omega(F)$ the zigzags containing the sequences $e, D_F(e)$ and $e', D_F(e')$ can be coincident. In such case the reversed zigzags also are coincident. □

The triangulation Γ is *locally z-knotted* for F if $|\mathcal{Z}(F)| = 2$. Thus, by Lemma 2, the triangulation Γ is locally z-knotted for F if and only if there is a single zigzag, up to reversing, containing edges of F. The following lemma can be proved in the similar way as Proposition 2.

Lemma 3. *If Γ is locally z-knotted for F, then every zigzag from $\mathcal{Z}(F)$ passes through each edge of F twice. Conversely, if there is a zigzag passing through each edge of F twice, then Γ is locally z-knotted for F.*

3.2 Classification of z-monodromies

Now, we present the classification of z-monodromies of faces in triangulations.

Theorem 2. *For the z-monodromy M_F one of the following possibilities is realized:*

(M1) *M_F is the identity,*

(M2) *$M_F = D_F$,*

(M3) *$M_F = (-e_1, e_2, e_3)(-e_3, -e_2, e_1)$,*

(M4) *$M_F = (e_1, -e_2)(e_2, -e_1)$, where e_3 and $-e_3$ are fixed points,* (M5) $M_F = (D_F)^{-1}$,

(M6) $M_F = (-e_1, e_3, e_2)(-e_2, -e_3, e_1)$,

(M7) $M_F = (e_1, e_2)(-e_2, -e_1)$, *where* e_3 *and* $-e_3$ *are fixed points*

where (e_1, e_2, e_3) *is one of the cycles in* D_F. *The triangulation* Γ *is locally* z*-knotted for* F *if and only if one of the cases* (M1)–(M4) *is realized.*

Later we give an example for every possibility presented in Theorem 2. Before we prove this theorem, we need the following two lemmas.

Lemma 4. *The following assertions are fulfilled:*

(1) *The equality* $M_F(e) = e'$ *implies that* $M_F(-e') = -e$.

(2) M_F *is bijective.*

(3) $M_F(e) \neq -e$ *for every* $e \in \Omega(F)$.

(4) *The length of every cycle in the permutation* M_F *is not greater than* 3.

Proof. (1). Let $e \in \Omega(F)$ and $e_0 \in \Omega(F)$ be such that $D_F(e_0) = e$. Let Z be the zigzag containing the sequence e_0, e. Thus

$$e' = M_F(e) \text{ and } e'_0 = D_F M_F(e)$$

are the next two elements of $\Omega(F)$ in Z. Note that $D_F(-e'_0) = -e'$. The reversed zigzag Z^{-1} contains the sequence $-e'_0, -e'$ and $-e$ is the first element of $\Omega(F)$ contained in Z^{-1} after $-e'$. Therefore, $M_F(-e') = -e$.

(2). It is sufficient to show that M_F is injective. If $M_F(e) = M_F(e') = e''$, then, by (1), we have $-e = M_F(-e'') = -e'$. This means that $e = e'$.

(3). Let e and e_0 be as in the proof of (1). Suppose that $M_F(e) = -e$. Then the zigzag containing the sequence e_0, e contains also the sequence $-e, D_F(-e)$. Since $D_F(-e) = -e_0$, this zigzag contains the sequence $-e, -e_0$ reversed to e_0, e, which is impossible.

(4). Suppose that the permutation M_F contains a cycle of the length greater than 3. Let $e_1, e_2, e_3 \in \Omega(F)$ be consecutive elements in this cycle. Then

$$M_F(e_1) = e_2, \quad M_F(e_2) = e_3, \quad M_F(e_3) \neq e_1$$

and the statement (3) implies $M_F(e_3) \neq -e_3$. Therefore, $M_F(e_3)$ is equal to $-e_1$ or $-e_2$. By (1), the equality $M_F(e_3) = -e_2$ implies that $M_F(e_2) = -e_3$. This is impossible, so $M_F(e_3) = -e_1$. Then $M_F(e_1) = -e_3$. The latter means that $e_2 = -e_3$ which contradicts $M_F(e_2) = e_3$ by (3). □

Lemma 5. *The triangulation Γ is locally z-knotted for F if and only if $D_F M_F$ is the composition of two distinct commuting 3-cycles.*

Proof. Let $e, e_0 \in \Omega(F)$ satisfy $D_F(e_0) = e$. We take the zigzag Z containing the sequence e_0, e. If Z contains the sequence e'_0, e', where $e', e'_0 \in \Omega(F)$ and $D_F(e'_0) = e'$, then we denote by $[e', M_F(e')]$ the part of Z between e' and $M_F(e')$. The zigzag Z is the cyclic sequence

$$[e, M_F(e)], [D_F M_F(e), M_F D_F M_F(e)], \dots, [(D_F M_F)^{m-1}(e), M_F(D_F M_F)^{m-1}(e)],$$

where m is the smallest non-zero number such that $(D_F M_F)^m(e) = e$. Since

$$e, D_F M_F(e), \dots, (D_F M_F)^{m-1}(e)$$

are mutually distinct, the same holds for

$$M_F(e), M_F D_F M_F(e), \dots, M_F(D_F M_F)^{m-1}(e).$$

Consider the following two sets:

$$\mathcal{X} = \{e, D_F M_F(e), \dots, (D_F M_F)^{m-1}(e)\},$$

$$\mathcal{Y} = \{-M_F(e), -M_F D_F M_F(e), \dots, -M_F(D_F M_F)^{m-1}(e)\}.$$

If e' is an element of $\mathcal{X} \cap \mathcal{Y}$ and $D_F(e'') = e'$, then $D_F(-e') = -e''$ and Z is a cyclic sequence of type

$$\dots, [*, e''], [e', *], \dots, [*, -e'], [-e'', *], \dots$$

Thus, Z is self-reversed, which is impossible, and $\mathcal{X} \cap \mathcal{Y} = \emptyset$. This implies that $m \leq 3$ because otherwise $\mathcal{X}$ and $\mathcal{Y}$ both contain more that three elements and have a non-empty intersection. The zigzag

$$Z = [e_1, M_F(e_1)], \dots, [e_m, M_F(e_m)], \quad \text{where} \quad e_i = (D_F M_F)^{i-1}(e),$$

corresponds to the m-cycle $C = (e_1, \dots, e_m)$ in the permutation $D_F M_F$. The reversed zigzag

$$Z^{-1} = [-M_F(e_m), -e_m], \dots, [-M_F(e_1), -e_1]$$

corresponds to the m-cycle

$$C' = (-M_F(e_m), \dots, -M_F(e_1))$$

If $m = 1$, then e_1 and $-M_F(e_1)$ are fixed points of $D_F M_F$.

Now, consider the case for $m = 3$. Then $D_F M_F$ is the composition of the commuting 3-cycles C and C'. Moreover $\Omega(F) = \mathcal{X} \cup \mathcal{Y}$ and every element of $\Omega(F)$ is contained in $\mathcal{X}$ or $\mathcal{Y}$. Therefore, $\mathcal{Z}(F) = \{Z, Z^{-1}\}$.

For $m < 3$ there are elements of $\Omega(F)$ which are not contained in $\mathcal{X} \cup \mathcal{Y}$. Such elements define zigzags distinct from Z and Z^{-1}. In this case $D_F M_F$ does not contain 3-cycles. $\square$

Proof of Theorem 2. We use Lemma 4 to show that M_F is one of the permutations (M1)–(M7). Assume that M_F is not identity.

Consider the case when M_F contains a 3-cycle C. The statement (3) from Lemma 4 guarantees that this cycle does not contain pairs of type $e, -e$. So, there exist $e_1, e_2, e_3 \in \Omega(F)$ such that (e_1, e_2, e_3) is a cycle in D_F and

$$C = (e_1, e_2, e_3) \;\text{ or }\; C = (-e_1, e_2, e_3) \;\text{ or }\; C = (e_1, e_3, e_2) \;\text{ or }\; C = (-e_1, e_3, e_2).$$

Using (1) from Lemma 4 we establish that

$$M_F = (e_1, e_2, e_3)(-e_3, -e_2, -e_1) \;\text{ or }\; M_F = (-e_1, e_2, e_3)(-e_3, -e_2, e_1)$$

or

$$M_F = (e_1, e_3, e_2)(-e_2, -e_3, -e_1) \;\text{ or }\; M_F = (-e_1, e_3, e_2)(-e_2, -e_3, e_1).$$

Therefore, we get permutations (M2), (M3), (M5) and (M6), respectively.

If M_F does not contain a 3-cycle, then it contains a transposition T. By the statement (3) from Lemma 4, the transposition T is not of type $(e, -e)$. Then there exist $e_1, e_2, e_3 \in \Omega(F)$ such that (e_1, e_2, e_3) is one of the cycles in D_F and

$$T = (e_1, e_2) \;\text{ or }\; T = (e_1, -e_2).$$

So, by (1) from Lemma 4,

$$M_F = (e_1, e_2)(-e_2, -e_1) \;\text{ or }\; M_F = (e_1, -e_2)(e_2, -e_1).$$

Then the statement (3) from Lemma 4 implies that e_3 and $-e_3$ are fixed points of M_F. Thus, we get (M7) or (M4), respectively.

A direct verification with the use of Lemma 5 shows that $D_F M_F$ is the composition of two distinct commuting 3-cycles if and only if M_F is one of (M1)–(M4). $\square$

Remark 2. There is a one-to-one correspondence between cycles of the permutation $D_F M_F$ and zigzags belonging to $\mathcal{Z}(F)$ (see the proof of Lemma 5). It is easy to verify that if M_F is (M5), then $|\mathcal{Z}(F)| = 6$ and if M_F is (M6) or (M7), then $|\mathcal{Z}(F)| = 4$.

3.3 A characterization of z-knotted triangulations

As a consequence of Theorem 2, we obtain the following characterization of z-knotted triangulations.

Theorem 3. *Γ is z-knotted if and only if for every face F the z-monodromy M_F is one of* (M1)–(M4).

Proof. If Γ is z-knotted, then it is locally z-knotted for each face F and, by Theorem 2, every z-monodromy M_F is one of (M1)–(M4).

Conversely, suppose that for every face F of Γ the z-monodromy M_F is one of (M1)–(M4). By Theorem 2, the triangulation Γ is locally z-knotted for all faces. If $\mathcal{Z}(F) = \{Z, Z^{-1}\}$, then Z passes through each edge of F. Therefore, if F' is a face adjacent to F, then they have a common edge and Z also belongs to $\mathcal{Z}(F')$ by (1) from Lemma 2. We have $\mathcal{Z}(F') = \{Z, Z^{-1}\}$ because Γ is locally z-knotted for F'. The same holds for all faces of Γ by connectedness. $\square$

3.4 Examples of z-monodromies

We give an example for each of types of z-monodromy from Theorem 2. For simplicity, all zigzags are written as sequences of vertices. The first four examples describe the z-monodromies (M3)–(M5) and (M7). Each of these z-monodromies is realized in a bipyramid. Since for any two faces in a bipyramid there is an automorphism transferring one of them to the other, the z-monodromies of all faces in each bipyramid are of the same type.

Example 8 (the z-monodromy of type (M3))**.** Suppose that $n = 2k + 1$ and k is odd. If $k = 1$, then one of the zigzags is

$$a, 1, 2, b, 3, 1, a, 2, 3, b, 1, 2, a, 3, 1, b, 2, 3.$$

For $k \geq 3$ this zigzag is

$$a, 1, 2, b, 3, 4, \dots, a, n-2, n-1, b, n, 1, a, 2, 3, b, \dots, a, n-1, n,$$

$$b, 1, 2, a, 3, 4, \dots, b, n-2, n-1, a, n, 1, b, 2, 3, a, \dots, b, n-1, n.$$

The zigzag passes through every edge twice, thus the bipyramid is z-knotted. The face F appears in the zigzag as follows

$$\dots, a, 1, 2, \dots, 1, a, 2, \dots, 1, 2, a, \dots$$

and it determines M_F for three elements of $\Omega(F)$

$$12 \to 1a, \quad a2 \to 12, \quad 2a \to a1.$$

By (1) from Lemma 4, we have

$$a1 \to 21, \quad 21 \to 2a, \quad 1a \to a2.$$

Let $e_1 = 12$, $e_2 = 2a$, $e_3 = a1$. Then (e_1, e_2, e_3) is one of the 3-cycles in D_F and

$$M_F = (-e_1, e_2, e_3)(-e_3, -e_2, e_1)$$

is of type (M3).

Example 9 (the z-monodromy of type (M4))**.** Suppose that $n = 2k + 1$ and k is even. If $k = 2$, then one of the zigzags is

$$a, 1, 2, b, 3, 4, a, 5, 1, b, 2, 3, a, 4, 5, b, 1, 2, a, 3, 4, b, 5, 1, a, 2, 3, b, 4, 5$$

and for $k \geq 4$ the zigzag is

$$a, 1, 2, b, 3, 4, a, \dots, b, n-2, n-1, a, n, 1, b, 2, 3, a, \dots, a, n-1, n,$$

$$b, 1, 2, a, 3, 4, b, \dots, a, n-2, n-1, b, n, 1, a, 2, 3, b, \dots, b, n-1, n.$$

As in the previous example, this zigzag passes through every edge twice and the bipyramid is z-knotted. Let $e_1 = 2a$, $e_2 = a1$, $e_3 = 12$. Then (e_1, e_2, e_3) is one of the 3-cycles in D_F and the face F appears in the zigzag as follows

$$\dots, a, 1, 2, \dots, 1, 2, a, \dots, 1, a, 2, \dots,$$

which implies that M_F leaves fixed e_3 and transfers e_1 to $-e_2$ and $-e_1$ to e_2. By the statement (1) from Lemma 4, we establish that

$$M_F = (e_1, -e_2)(e_2, -e_1)$$

is of type (M4).

Example 10 (the z-monodromy of type (M5))**.** Suppose that $n = 2k$ and k is even. In the case when $k = 2$, we get the octahedron whose zigzags are vertex-simple. Assume that $k \geq 4$. The set $\mathcal{Z}(F)$ contains precisely 8 zigzags:

- $a, 1, 2, b, 3, 4, \dots, b, n-1, n$

- $b, 1, 2, a, 3, 4, \ldots, a, n-1, n$
- $1, a, 2, 3, b, \ldots, a, n-2, n-1, b, n$
- $1, b, 2, 3, a, \ldots, b, n-2, n-1, a, n$

and their reverses. These zigzags are not vertex-simple, but they are edge-simple. It means that $M_F(e) = (D_F)^{-1}$ for any face F in the bipyramid and for any $e \in \Omega(F)$. Therefore, the z-monodromy of every face is of type (M5).

Example 11 (the z-monodromy of type (M7))**.** Suppose that $n = 2k$ and k is an odd number greater than 1. The set $\mathcal{Z}(F)$ consists of the following two zigzags

$$a, 1, 2, b, 3, 4, \ldots, a, n-1, n, b, 1, 2, a, 3, 4, \ldots, b, n-1, n$$

and

$$1, a, 2, 3, \ldots, b, n-2, n-1, a, n, 1, b, 2, 3, \ldots, a, n-2, n-1, b, n$$

and their reverses. We take $e_1 = 2a$, $e_2 = a1$, $e_3 = 12$ and then (e_1, e_2, e_3) is one of the 3-cycles in D_F. The face F appears in the zigzags as follows

$$\ldots, a, 1, 2, \ldots, 1, 2, a, \ldots \quad \text{and} \quad \ldots, 1, a, 2, \ldots$$

which implies that M_F leaves fixed e_3 and transfers e_1 to e_2 and $-e_1$ to $-e_2$. By (1) from Lemma 4

$$M_F = (e_1, e_2)(-e_2, -e_1)$$

is of type (M7) and z-monodromies of all faces in the bipyramid are of this type.

The next three examples are connected sums of bipyramids with z-monodromies (M1), (M2) and (M6). Let BP_n be as previous and, similarly, let $BP_{n'}$ be bipyramid with the base whose vertices are $1', \ldots, n'$ and two remaining vertices denoted by a', b'. Let F_1 and F_2 be the faces of the bipyramids containing the vertices $a, 1, 2$ and $a', 1', 2'$, respectively, and $g : \partial F_1 \to \partial F_2$ be a special homeomorphism.

Example 12 (the z-monodromy of type (M2))**.** Suppose that $n = 2k + 1$ and $n' = 2k' + 1$, where k and k' are odd. Denote by Γ the connected sum $BP_n \#_g BP_{n'}$, where $g : \partial F_1 \to \partial F_2$ satisfies

$$g(a) = a', \;\; g(1) = 1', \;\; g(2) = 2'.$$

The zigzag of BP_n considered in Example 8 can be presented as the cyclic sequence A, B, C, where

$$A = \{1, 2, b, \ldots, 1, a\}, \;\; B = \{a, 2, \ldots, b, 1, 2\}, \;\; C = \{2, a, \ldots, 1, b, 2, \ldots, a, 1\}$$

are parts of the zigzag between two edges of the face F_1. Note that any two consecutive parts have the same vertex (for example, the parts A and B are joined in the vertex a). Similarly, one of the two zigzags of $BP_{n'}$ is the cyclic sequence A', B', C', where

$$A' = \{1', 2', \dots, 1', a'\}, \quad B' = \{a', 2', \dots, 1', 2'\}, \quad C' = \{2', a', \dots, a', 1'\}.$$

Consider the cyclic sequence

$$A, C'^{-1}, B, A', C^{-1}, B',$$

where for any two consecutive parts X, Y the last edge from X is identified with the first edge from Y. A direct verification shows that this is a zigzag in Γ. This zigzag passes through each edge of Γ twice (since it is obtained from a zigzag of BP_n passing through all edges of BP_n twice and a zigzag of $BP_{n'}$ satisfying the same condition). Therefore, Γ is z-knotted. Let F be the face of Γ containing the vertices $b, 1, 2$ and let $e_1 = 12$, $e_2 = 2b$, $e_3 = b1$. Since this face appears in the zigzag as follows

$$\dots, 1, 2, b, \dots, b, 1, 2, \dots, 2, b, 1, \dots,$$

the z-monodromy M_F contains the 3-cycle (e_1, e_2, e_3). By the statement (1) from Lemma 4,

$$M_F = (e_1, e_2, e_3)(-e_3, -e_2, -e_1) = D_F$$

is of type (M2).

Example 13 (the z-monodromy of type (M1))**.** Suppose that $n = 2k$ and $n' = 2k'$, where k and k' are odd numbers greater than 1. Denote by Γ the connected sum $BP_n \#_g BP_{n'}$, where $g : \partial F_1 \to \partial F_2$ satisfies

$$g(a) = 2', \quad g(1) = a', \quad g(2) = 1'.$$

By Example 11, the set $\mathcal{Z}(F_1)$ is formed by the zigzags

$$a, 1, 2, b, 3, 4, \dots, a, n-1, n, b, 1, 2, a, 3, 4, \dots, b, n-1, n$$

and

$$1, a, 2, 3, \dots, b, n-2, n-1, a, n, 1, b, 2, 3, \dots, a, n-2, n-1, b, n$$

and their reverses. The first zigzag is the cyclic sequence A, B, where

$$A = \{1, 2, \dots, 1, 2\} \text{ and } B = \{2, a, 3, \dots, n, a, 1\}$$

are parts of the zigzag between two edges of the face F_1 (A is joined with B in the vertex 2 and B is joined with A in the vertex 1). The second zigzag passes once through the edges $a1$ and $a2$, and it does not contain the edge 12. We rewrite this zigzag as

$$C = \{a, 2, 3, \dots, 2, 3, a, \dots, n, 1, a\}.$$

Similarly, $\mathcal{Z}(F_2)$ consists of four zigzags: one of them is A', B', the second is C', where

$$A' = \{1', 2', \dots, 1', 2'\}, \quad B' = \{2', a', \dots, a', 1'\}, \quad C' = \{a', 2', \dots, 1', a'\},$$

and the remaining two zigzags are their reverses. Then

$$A, C'^{-1}, C^{-1}, A', B, B'$$

(as in the previous example, for any two consecutive parts X, Y the last edge from X is identified with the first edge from Y) is a zigzag in Γ. This zigzag passes through all edges of Γ twice (indeed, the sequence A, B, C contains every edge of BP_n twice and A', B', C' contains every edge of $BP_{n'}$ twice). Therefore, Γ is z-knotted. Let F be the face of Γ containing the vertices $a, 2, 3$. This face appears in the zigzag as follows

$$\dots, a, 3, 2, \dots, 3, 2, a, \dots, 2, a, 3, \dots$$

which implies that M_F is identity.

Example 14 (the z-monodromy of type (M6))**.** As in Example 12, we suppose that $n = 2k + 1$ and $n' = 2k' + 1$, where k and k' are odd. Consider the connected sum $BP_n \#_g BP_{n'}$, where $g : \partial F_1 \to \partial F_2$ satisfies

$$g(a) = 1', \quad g(1) = 2', \quad g(2) = a'.$$

Recall that the single zigzags (up to reversing) in BP_n and $BP_{n'}$ are the cyclic sequences A, B, C and A', B', C' (respectively), where

$$A = \{1, 2, b, \dots, 1, a\}, \quad B = \{a, 2, \dots, b, 1, 2\}, \quad C = \{2, a, \dots, 1, b, 2, \dots, a, 1\},$$

$$A' = \{1', 2', \dots, 1', a'\}, \quad B' = \{a', 2', \dots, 1', 2', \}, \quad C' = \{2', a', \dots, a', 1'\}.$$

The cyclic sequences

$$A, B'^{-1} \text{ and } B, C', C, A'$$

define zigzags in $BP_n \#_g BP_{n'}$ (the corresponding edges in consecutive parts are identified). Let F be the face of the connected sum which contains the vertices $b, 1, 2$

and let $e_1 = b1$, $e_2 = 12$, $e_3 = 2b$. Then (e_1, e_2, e_3) is one of the 3-cycles in D_F. The face F appears in the zigzags as follows

$$\ldots, 1, 2, b, \ldots \quad \text{and} \quad \ldots, b, 1, 2, \ldots, 1, b, 2, \ldots$$

which determines M_F on three elements of $\Omega(F)$

$$e_3 \to e_2, \quad e_2 \to -e_1, \quad -e_3 \to e_1.$$

The statement (1) from Lemma 4 implies that

$$-e_2 \to -e_3, \quad e_1 \to -e_2, \quad -e_1 \to e_3.$$

Therefore,

$$M_F = (-e_1, e_3, e_2)(-e_2, -e_3, e_1)$$

is of type (M6).

4 Zigzags in connected sums of triangulations

4.1 Z-knotted shreddings

As before, we suppose that Γ is a triangulation of a surface S. Assume that $F_1, \dots, F_k$ are mutually distinct faces of Γ. Let $\Gamma_1, \dots, \Gamma_k$ be triangulations of a sphere $\mathbb{S}$. We take a face F'_i in Γ_i and a special homeomorphism $g_i : \partial F_i \to \partial F'_i$ for every $i \in \{1, \dots, k\}$. Then the connected sum

$$(((\Gamma \#_{g_1} \Gamma_1) \#_{g_2} \Gamma_2) \dots) \#_{g_k} \Gamma_k \tag{1}$$

is a triangulation of S. In other words, the connected sum (1) is obtained from Γ by replacing every F_i by a triangulation of a 2-dimensional disc. Every triangulation of S obtained from Γ in such a way is called a *shredding* of Γ.

The main result of this section is the following theorem.

Theorem 4 ([25])**.** *Every triangulation Γ of any connected closed 2-dimensional surface admits a z-knotted shredding. Suppose that Γ contains precisely $2m$ zigzags, i.e. m zigzags up to reversing, and $m > 1$. Then there are z-knotted triangulations $\Gamma_1, \dots, \Gamma_k$ of a sphere $\mathbb{S}$ such that $k \leq m - 1$ and the connected sum* (1) *is z-knotted.*

4.2 Proof of Theorem 4

The crucial tools used to prove Theorem 4 are our classification of z-monodromies (Theorem 2) and the gluing lemma (Lemma 6).

Let F and F' be faces in triangulations Γ and Γ' (respectively). Let $g : \partial F \to \partial F'$ be a special homeomorphism. It induces a bijection between $\Omega(F)$ and $\Omega(F')$ which transfers each oriented edge xy to the oriented edge $g(x)g(y)$. We will denote this bijection also by g. Observe that the bijection $g : \Omega(F) \to \Omega(F')$ has the following two properties:

- $g(-e) = -g(e)$ for every $e \in \Omega(F)$,
- $gD_F g^{-1} = D_{F'}$.

A face D in a triangulation is said to be *essential* if every zigzag of this triangulation contains an edge from this face (in other words, every zigzag belongs to $\mathcal{Z}(D)$). For example, all faces of z-knotted triangulations and all faces of a tetrahedron are essential.

Lemma 6 (Gluing lemma)**.** *The following assertions are fulfilled:*

(1) *Suppose that F and F' are essential faces. Then the connected sum $\Gamma\#_g\Gamma'$ is z-knotted if and only if $gM_Fg^{-1}M_{F'}$ is the composition of two distinct commuting 3-cycles.*

(2) *Suppose that Γ' is z-knotted and $gM_Fg^{-1}M_{F'}$ is the composition of two distinct commuting 3-cycles. Then $\Gamma\#_g\Gamma'$ contains a zigzag Z such that $\mathcal{Z}(D)=\{Z,Z^{-1}\}$ for every face D in $\Gamma\#_g\Gamma'$ corresponding to a face of Γ' distinct from F'.*

Proof. There exists a zigzag of Γ containing the sequence $(D_F)^{-1}(e), e$ and $M_F(e)$ for every $e\in\Omega(F)$, see the definition of z-monodromy. We write $[e,M_F(e)]$ for the part of this zigzag between e and $M_F(e)$ (as in the proof of Lemma 5) and denote by $\mathcal{X}$ the set of all such parts for $e\in\Omega(F)$. Note that for $X=[e,M_F(e)]$ contained in $\mathcal{X}$ its reversed path $X^{-1}=[e',M_F(e')]$ (where $e'=-M_F(e)$) also belongs to $\mathcal{X}$ and $\mathcal{X}$ contains 6 elements. In the similar way, we introduce $\mathcal{X}'$ as the set of all $[e,M_{F'}(e)]$ for F' in Γ', where $e\in\Omega(F')$.

Consider the following cyclic sequence for $e\in\Omega(F')$

$$[e,M_{F'}(e)],[g^{-1}M_{F'}(e),M_Fg^{-1}M_{F'}(e)],[gM_Fg^{-1}M_{F'}(e),M_{F'}(gM_Fg^{-1}M_{F'})(e)],$$
$$\vdots$$
$$[g^{-1}M_{F'}(gM_Fg^{-1}M_{F'})^{m-1}(e),M_Fg^{-1}M_{F'}(gM_Fg^{-1}M_{F'})^{m-1}(e)],$$

where m is the smallest positive number such that

$$(gM_Fg^{-1}M_{F'})^m(e)=e.$$

So, this is a cyclic sequence

$$X'_1,X_1,\dots,X'_m,X_m,$$

where $X_i\in\mathcal{X}$ and $X'_i\in\mathcal{X}'$ for all $i\in\{1,\dots,m\}$. For any two consecutive parts $A,B\in\mathcal{X}\cup\mathcal{X}'$ from the above sequence we identify the last edge from A with the first edge from B. As the result, we get a zigzag in the connected sum $\Gamma\#_g\Gamma'$ and denote it by $Z(e)$. Zigzags are not self-reversed (by Proposition 1), thus

$$X_i\neq X_j^{-1}\ \text{ and }\ X'_i\neq X'^{-1}_j$$

for any pair $i,j\in\{1,\dots,m\}$ and, as consequence, $m\leq 3$.

The zigzag $Z(e)$ is related to a cycle of length m in the permutation $gM_Fg^{-1}M_{F'}$. The reversed zigzag $Z(e)^{-1}$ is the cyclic sequence

$$X_m^{-1},X'^{-1}_m,\dots,X_1^{-1},X'^{-1}_1$$

corresponding to another cycle of length m in this permutation. Observe that $Z(e)^{-1} = Z(e')$ for a certain $e' \in \Omega(F')$. As in the proof of Lemma 5, we establish that the following conditions are equivalent:

(A) the permutation $gM_F g^{-1} M_{F'}$ is the composition of two distinct commuting 3-cycles,

(B) for any $e, e' \in \Omega(F')$ the zigzag $Z(e')$ coincides with $Z(e)$ or $Z(e)^{-1}$.

It remains to prove the statements (1) and (2).

(1). If the faces F and F' are essential, then every zigzag of Γ or Γ' contains an edge from F or F', respectively. Therefore, each of zigzags in $\Gamma \#_g \Gamma'$ is of type $Z(e)$, where $e \in \Omega(F')$. In such case, the condition (B) is equivalent to the z-knottedness of $\Gamma \#_g \Gamma'$.

(2). Assume that Γ' is z-knotted and the condition (A) is fulfilled. Then (B) is also fulfilled and we establish that the zigzag $Z = Z(e)$ (where $e \in \Omega(F')$) is as required.

Let Z', Z'^{-1} be a single pair of zigzags in Γ'. As in the proof of Lemma 5, we have

$$Z' = X'_1, X'_2, X'_3 \ \text{ and } \ Z'^{-1} = X'^{-1}_3, X'^{-1}_2, X'^{-1}_1,$$

where $X'_1, X'_2, X'_3, X'^{-1}_1, X'^{-1}_2, X'^{-1}_3$ are all mutually distinct elements of $\mathcal{X}'$. By the condition (B) every $X' \in \mathcal{X}'$ is contained in Z or Z^{-1}.

Corollary 1 guarantees that Z' and Z'^{-1} passes through each of edges of Γ' twice. Thus the zigzags Z and Z^{-1} also passes through each of edges of Γ' twice and there is no other zigzags in $\Gamma \#_g \Gamma'$ containing edges from Γ'. Therefore, $\mathcal{Z}(D) = \{Z, Z^{-1}\}$ for every face $D \neq F'$ of Γ' considered as a face of the connected sum $\Gamma \#_g \Gamma'$. $\square$

From this moment, we will suppose that Γ is a triangulation which is not locally z-knotted for a certain face F.

Lemma 7. *There is a z-knotted triangulation Γ' of a sphere $\mathbb{S}$ and a face F' in this triangulation such that $gM_F g^{-1} M_{F'}$ is the composition of two distinct commuting 3-cycles for a certain special homeomorphism $g : \partial F \to \partial F'$.*

Proof. It follows from Theorem 2 that the z-monodromy M_F is one of (M5)–(M7).

Consider the case when M_F is (M5) or (M6). So, it is the composition of two distinct commuting 3-cycles. We take a z-knotted triangulation Γ' of a sphere containing a face F' such that $M_{F'}$ is identity (for example, the triangulation from Example 13). Then, the permutation

$$gM_F g^{-1} M_{F'} = gM_F g^{-1}$$

is the composition of two distinct commuting 3-cycles for every special homeomorphism $g : \partial F \to \partial F'$.

Now, let M_F be of type (M7), i.e.

$$M_F = (e_1, e_2)(-e_1, -e_2),$$

where (e_1, e_2, e_3) is one of the cycles in D_F and $e_3, -e_3$ are fixed points. Suppose that Γ' is the bipyramid BP_{2k+1}, where k is odd. For every face F' of Γ' its z-monodromy is

$$M_{F'} = (-e'_1, e'_2, e'_3)(-e'_3, -e'_2, e'_1),$$

where (e'_1, e'_2, e'_3) is one of the cycles in the permutation $D_{F'}$ (see Example 8). It is easy to verify that $gM_Fg^{-1}M_{F'}$ is the composition of two distinct commuting 3-cycles for a special homeomorphism $g : \partial F \to \partial F'$ such that $g(e_i) = e'_i$ for every i. □

Let F' be a face in a z-knotted triangulation Γ' of a sphere and $g : \partial F \to \partial F'$ be a special homeomorphism such that $gM_Fg^{-1}M_{F'}$ is the composition of two distinct commuting 3-cycles (as in Lemma 7). By the statement (2) from Lemma 6, the connected sum $\Gamma\#_g\Gamma'$ contains a zigzag Z such that $\mathcal{Z}(D) = \{Z, Z^{-1}\}$ for every face D in $\Gamma\#_g\Gamma'$ corresponding to a face of Γ' distinct from F'. In other words, $\Gamma\#_g\Gamma'$ is locally z-knotted for all faces of Γ' distinct from F'.

Lemma 8. *Suppose that D is a face of Γ distinct from F. If Γ is locally z-knotted for D, then $\Gamma\#_g\Gamma'$ also is locally z-knotted for D.*

Proof. There is a unique pair of zigzags in Γ containing edges of D. Denote these zigzags by Z_D and $(Z_D)^{-1}$. Each of them passes through every edge of D twice (by Lemma 3).

If Z_D and $(Z_D)^{-1}$ do not contain any edge from F, then they are also zigzags in $\Gamma\#_g\Gamma'$ and the connected sum is locally z-knotted for D.

If there are edges of F contained in Z_D or $(Z_D)^{-1}$, then these zigzags belong to $\mathcal{Z}(F)$. As in the proof of Lemma 6, we introduce the set $\mathcal{X}$ consisting of all parts of the zigzags from $\mathcal{Z}(F)$ between $e \in \Omega(F)$ and $M_F(e)$. We remind that every $X \in \mathcal{X}$ is contained in Z or Z^{-1}. The zigzags Z_D and $(Z_D)^{-1}$ belong to $\mathcal{Z}(F)$, so they are cyclic sequences formed by at most three elements of $\mathcal{X}$. Each of these zigzags passes through every edge of D twice and the same holds for Z and Z^{-1}. Lemma 3 implies that $\Gamma\#_g\Gamma'$ is locally z-knotted for D. □

Now, we complete the proof of Theorem 4. It was noted before Lemma 8 that

(1) $\Gamma\#_g\Gamma'$ is locally z-knotted for all faces of Γ' distinct from F'.

By Lemma 8, we have that

(2) if Γ is locally z-knotted for a face $D \neq F$, then $\Gamma \#_g \Gamma'$ is locally z-knotted for D.

It follows from (1) that if the connected sum $\Gamma \#_g \Gamma'$ is not locally z-knotted for a face T, then T is a face of Γ. We apply the above arguments to the face T in $\Gamma \#_g \Gamma'$. By (2), we can construct recursively a shredding of Γ that is locally z-knotted for each face. This shredding is z-knotted by Theorem 3.

Assume that Γ contains precisely $2m$ zigzags, i.e. m zigzags up to reversing. If Γ is not locally z-knotted for a face F, then $\mathcal{Z}(F)$ includes 4 or 6 zigzags (see Remark 2). We replace these zigzags with one zigzag using the corresponding connected sum and come to a triangulation with $m-1$ or $m-2$ zigzags up to reversing. So, to produce a z-knotted shredding of Γ we need at most $m-1$ times. This completes the proof of Theorem 4.

4.3 Application to connected sums of z-knotted triangulations

According to Theorem 2, there are precisely four types of faces in z-knotted triangulations and their z-monodromies are (M1)–(M4). These four types were described without z-monodromies in [23]. The main result of [23] presents all cases when the connected sum of two z-knotted triangulations is z-knotted. The proof given in [23] is a long case-by-case analysis. Following [25] we obtain this result as a consequence of the statement (1) from Lemma 6.

Theorem 5 ([23, 25]). *Let Γ and Γ' be z-knotted triangulations. Then the following assertions are fulfilled:*

(1) *If F is a face in Γ such that $M_F = D_F$ (type (M2)), then for every face F' in Γ' and every special homeomorphism $g : \partial F \to \partial F'$ the connected sum $\Gamma \#_g \Gamma'$ is z-knotted.*

(2) *Suppose that F is a face in Γ and M_F is identity (type (M1)). If F' is a face in Γ' such that the connected sum $\Gamma \#_g \Gamma'$ is z-knotted for a certain special homeomorphism $g : \partial F \to \partial F'$, then $M_{F'}$ is $D_{F'}$ or (M3). In these cases, the connected sum $\Gamma \#_g \Gamma'$ is z-knotted for every special homeomorphism $g : \partial F \to \partial F'$.*

(3) *If F is a face in Γ, F' is a face in Γ' and $M_F, M_{F'}$ are (M3) or (M4), then there is a special homeomorphism $g : \partial F \to \partial F'$ such that the connected sum $\Gamma \#_g \Gamma'$ is z-knotted.*

Proof. Remind that any special homeomorphism $g : \partial F \to \partial F'$ gives a bijection $\Omega(F)$ to $\Omega(F')$, which is also denoted by g.

(1). It is clear that $gD_Fg^{-1} = D_{F'}$. So, $M_F = D_F$ implies

$$gM_Fg^{-1}M_{F'} = D_{F'}M_{F'}$$

for each special homeomorphism $g : \partial F \to \partial F'$. Using z-knottedness of Γ' and Lemma 5, we conclude that $D_{F'}M_{F'}$ is the composition of two distinct commuting 3-cycles. Thus, the connected sum $\Gamma\#_g\Gamma'$ is z-knotted by the statement (1) from Lemma 6.

(2). Consider the case when M_F is identity, i.e. M_F is of type (M1). Then

$$gM_Fg^{-1}M_{F'} = M_{F'}$$

for every special homeomorphism $g : \partial F \to \partial F'$. The above is the composition of two distinct commuting 3-cycles precisely when $M_{F'}$ is $D_{F'}$ or of type (M3). The statement (1) from Lemma 6 gives the claim.

(3). Let M_F and $M_{F'}$ be of type (M3), i.e.

$$M_F = (-e_1, e_2, e_3)(-e_3, -e_2, e_1) \text{ and } M_{F'} = (-e'_1, e'_2, e'_3)(-e'_3, -e'_2, e'_1),$$

where (e_1, e_2, e_3) and (e'_1, e'_2, e'_3) are 3-cycles in D_F and $D_{F'}$, respectively. We take a special homeomorphism $g : \partial F \to \partial F'$ sending e_1, e_2, e_3 to e'_1, e'_2, e'_3, respectively. Then $gM_Fg^{-1} = M_{F'}$ and

$$gM_Fg^{-1}M_{F'} = (M_{F'})^2 = (M_{F'})^{-1}$$

is the composition of two distinct commuting 3-cycles.

Assume that F and F' are of type (M4), i.e.

$$M_F = (e_1, -e_2)(e_2, -e_1) \text{ and } M_{F'} = (e'_1, -e'_2)(e'_2, -e'_1),$$

where (e_1, e_2, e_3) and (e'_1, e'_2, e'_3) are 3-cycles in D_F and $D_{F'}$, respectively. Using a special homeomorphism $g : \partial F \to \partial F'$, which transfers e_1, e_2, e_3 to e'_3, e'_1, e'_2 (respectively), we obtain

$$gM_Fg^{-1} = (e'_1, -e'_3)(e'_3, -e'_1).$$

Then

$$gM_Fg^{-1}M_{F'} = (-e'_1, e'_2, e'_3)(e'_1, -e'_2, -e'_3)$$

is the composition of two distinct commuting 3-cycles.

Let F be of type (M4) and F' be of type (M3), i.e.

$$M_F = (e_1, -e_2)(e_2, -e_1) \text{ and } M_{F'} = (-e'_1, e'_2, e'_3)(-e'_3, -e'_2, e'_1),$$

where (e_1, e_2, e_3) and (e'_1, e'_2, e'_3) are 3-cycles in D_F and $D_{F'}$, respectively. We take a special homeomorphism $g : \partial F \to \partial F'$ transferring e_1, e_2, e_3 to e'_2, e'_3, e'_1 (respectively). Then, we obtain

$$gM_F g^{-1} = (e'_2, -e'_3)(e'_3, -e'_2)$$

and

$$gM_F g^{-1} M_{F'} = (e'_1, e'_2, -e'_2)(-e'_1, -e'_3, e'_3)$$

is the composition of two distinct commuting 3-cycles.

The connected sum $\Gamma \#_g \Gamma'$ is z-knotted by the statement (1) from Lemma 6 in all of the above three cases. □

5 Z-oriented triangulations

In this section we work with z-oriented triangulations. In Subsection 2.3 we proved that there are two types of edges and faces in such triangulations. We recall that the choose of a z-orientation determines the directions on edges of type II and the subgraph Γ_{II} formed by these edges is considered as a digraph. At the beginning, we show that the general case of z-oriented triangulations can be reduced to the case where all faces are of type I. All results of this section come from [31].

5.1 Reduction to the case when all faces of type I

Proposition 4. *Any z-oriented triangulation (Γ, τ) admits a z-oriented shredding (Γ', τ') where all faces are of type I and $\Gamma_{II} = \Gamma'_{II}$.*

Proof. Suppose that F is a face of type II in (Γ, τ). Assume e_1, e_2, e_3 are edges of F and they are oriented as in Fig. 12. Let σ be the permutation $(1, 2, 3)$.

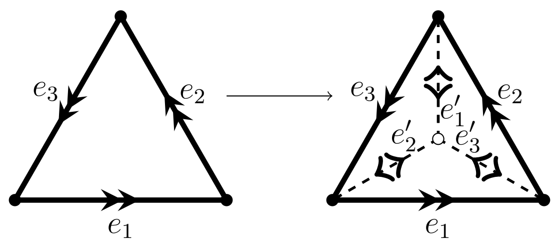

Figure 12

Zigzags from τ pass through F three times, thus the face F splits them into three segments of type

$$e_{\sigma^{-1}(i)}, e_i, X_{ij}, e_j, e_{\sigma(j)},$$

where $i, j \in \{1, 2, 3\}$, and the sequence X_{ij} is a maximal component of a zigzag consisting of edges occurring between e_i and e_j. Let $\mathcal{X}$ be the set of all such sequences X_{ij} obtained for the face F and the z-orientation τ. Observe that for each pair e_i, e_j, there exists a unique sequence X_{ij} such that e_i is directly before X_{ij} and e_j is directly after X_{ij} in one of zigzags. Now, we triangulate the face F by attaching a vertex in its interior and three edges joining this vertex with the vertices of F, see Fig. 12. Let this new triangulation be denoted by Γ' and let e'_i be the one of the new edges which does not has a common vertex with e_i. Note that for any $i \in \{1, 2, 3\}$ there exists a zigzag in Γ' containing a subsequence of the following form

$$e_i, e'_{\sigma^{-1}(i)}, e'_i, e_{\sigma^{-1}(i)}, X_{\sigma^{-1}(i)j}$$

for certain $j \in \{1, 2, 3\}$ and $X_{\sigma^{-1}(i)j} \in \mathcal{X}$. Since the edges of F are not included in $X_{\sigma^{-1}(i)j}$, the edge e_j occurring in the zigzag directly after this subsequence is the same as the edge after $X_{\sigma^{-1}(i)j}$ in (Γ, τ). Thus, zigzags of Γ' crossing any of the three new faces pass through the edges coming from Γ in the same way as zigzags from τ. So, there exists a z-orientation of Γ' which does not change types of edges from Γ. The three new faces of Γ', obtained by partition of F, are of type I for this z-orientation. Step by step, we replace all faces of type II from (Γ, τ) by faces of type I and get a z-oriented shredding of Γ with all faces of type I and unchanged types of edges coming from (Γ, τ). $\square$

5.2 Homogeneous zigzags in triangulations with faces of type I

Let Γ be a triangulation of a surface S. Suppose that there is a z-orientation τ of Γ such that all faces are of type I.

If m is the number of faces, then there are precisely m edges of type I and $m/2$ edges of type II, i.e. the number of edges of type I is twice the number of edges of type II. We say that a zigzag of (Γ, τ) is *homogeneous* if it is a cyclic sequence $\{e_i, e'_i, e''_i\}_{i=1}^{n}$, where each e_i is an edge of type II and all e'_i, e''_i are edges of type I. If Z is a homogeneous zigzag, then Z^{-1} also is homogeneous. We say that (Γ, τ) is *z-homogeneous* if all its zigzags are homogeneous.

Example 15. Let us return to Example 7 and suppose that $\Gamma = BP_n$. If n is odd, then Γ is a z-knotted bipyramid and it is z-homogeneous for both z-orientations. If n is even, then Γ is z-homogeneous for the z-orientation consisting of the two zigzags from Example 6 (b) or the four zigzags from Example 6 (c). The vertices a and b are of type I and the remaining are of type II. The subgraph Γ_{II} is the base of BP_n and it is the directed cycle. Conversely, if a triangulation is z-homogeneous for a certain z-orientation and there are precisely two vertices of type I, then this triangulation is a bipyramid. This fact is a direct consequence of Theorem 6 which will be presented later.

Example 16. Let Γ' be a triangulation with a z-orientation such that all faces are of type II. We consider Γ'' which is the shredding of Γ' constructed by adding a vertex in the interior of each face and three edges joining this vertex with the vertices of the face (as in the proof of Proposition 4). We obtain a z-orientation τ'' of Γ'' such that all its faces are of type I and each zigzag $\ldots, e_1, e_2, e_3, \ldots$ in Γ' is extended to a zigzag

$$\ldots, e_1, e'_1, e''_1, e_2, e'_2, e''_2, e_3, \ldots$$

in Γ''. Observe that both these zigzags pass through edges of Γ' in different directions. All e_i are of type II and all e'_i and e''_i are of type I. Thus, Γ'' is z-homogeneous for this z-orientation.

An *Eulerian digraph* is a connected digraph such that there exists a closed trail containing all directed edges of the digraph (see, for example, [33, p. 105]).

Theorem 6. *The following three conditions are equivalent:*

(1) (Γ, τ) *is z-homogeneous.*

(2) Γ_{II} *is a closed 2-cell embedding of a simple Eulerian digraph such that the edges of every face form a directed cycle.*

(3) *Each connected component of $S \setminus \Gamma_{II}$ is homeomorphic to an open 2-dimensional disc.*

The implication (2) $\Rightarrow$ (3) is clear. In the next two subsections we will prove the implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1).

5.3 Proof of the implication (1) $\Rightarrow$ (2) in Theorem 6

As in the previous subsection, (Γ, τ) is a z-oriented triangulation where all faces are of type I.

The construction from Proposition 4 and Example 16 can be generalized as follows. Let Γ' be a closed 2-cell embedding of a connected finite simple graph in S. Thus, all faces of Γ' are homeomorphic to a closed 2-dimensional disc. For each face F, we take a point v_F from the interior of F and we add v_F to the vertex set of Γ'. Then we join each v_F with each vertex of F by an edge. This new triangulation of S is denoted by $\mathrm{T}(\Gamma')$.

Note that if a certain face F of Γ' is not homeomorphic to a closed 2-dimensional disc, then v_F and one of vertices of F are connected by a double edge in $\mathrm{T}(\Gamma')$. It does not meet our definition of triangulation, so we need to assume that Γ' is a closed 2-cell embedding.

Proposition 5. *The following assertions are fulfilled:*

(I) *If (Γ, τ) is z-homogeneous, then Γ_{II} is a closed 2-cell embedding of a simple Eulerian digraph such that the edges of every face form a directed cycle and $\Gamma = \mathrm{T}(\Gamma_{II})$.*

(II) *Conversely, if* Γ' *is a closed* 2*-cell embedding of a simple Eulerian digraph and the edges of every face form a directed cycle, then there is a unique* z*-orientation of* $\mathrm{T}(\Gamma')$ *such that it is* z*-homogeneous triangulation and* Γ' *is the subgraph of* $\mathrm{T}(\Gamma')$ *formed by all vertices and edges of type II.*

Proof. (I). Assume v is a vertex of Γ. Consider all faces that contain v and take the edge from each of these faces that does not contain v. We denote a cycle formed by these edges by $C(v)$.

Suppose that (Γ, τ) is z-homogeneous. Consider any edge e_1 of type II and denote by v_1 and v_2 the vertices of this edge so that e_1 is directed from v_1 to v_2. We choose one of the two faces which contain e_1 and we denote by v the third vertex of this face (this vertex does not belong to e_1). Let e_1' and e_1'' be the edges containing v and which are immediately after e_1 in a certain zigzag Z from τ (see Fig. 13). The third edge of the face containing e_1' and e_1'' will be denoted by e_2. The edge e_2 consists of v_2 and another vertex, say v_3. The edges e_1' and e_1'' are of type I by homogeneity of the zigzag Z, so e_2 is of type II. Since Z goes through e_1' from v_2 to v and e_1' is an edge of type I, the zigzag going through e_1' in the opposite direction belongs to τ. Thus, the edge e_2 is directed from v_2 to v_3. The edge e_3 occurring in Z immediately after e_1' and e_1'' is of type II and it is directed from v_3 to a certain vertex v_4. So, the edges e_1, e_2, e_3 are consecutive in the cycle $C(v)$ and each e_i is directed from v_i to v_{i+1}. Consider the zigzag from τ which contains the sequence e_2, e_1''. This zigzag passes through the next edge from v to v_4. Let e_4 be the edge which occurs in the zigzag after it. By the assumption, e_4 is of type II. This is an edge of $C(v)$ which leaves the vertex v_4. Recursively, we establish that $C(v)$ is a directed cycle formed by edges of type II and v is a vertex of type I.

Now, we take the other face which contains e_1. Let v' be the vertex from this face other than v_1 and v_2. We repeat the above arguments and we establish that v' is of type I and $C(v')$ is a directed cycle whose all edges are of type II.

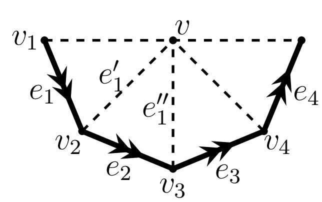

Figure 13

For every vertex v of type I there can be taken a face which contains v and the edge of this face which not containing v. Since the remaining two edges of the face are of type I, this edge is of type II. Using the above arguments, we establish that:

(1) there exist vertices of type I and the cycle $C(v)$ is a directed cycle formed by edges of type II for every vertex v of type I;

(2) for every edge of type II there are precisely two vertices v and v' of type I such that this edge is contained in $C(v)$ and $C(v')$.

In a similar way, for every edge e of type I we can take a face whose one of edges is e. Since this face contains an edge of type II, the vertices of e are of different types.

Since for any two vertices of type II in Γ we can find a path formed by edges of type II, the digraph Γ_{II} is connected. Otherwise, i.e. when a path between two vertices of type II goes through a vertex v of type I, the edge which enters to v and the edge which leaves v contain vertices of the same cycle $C(v)$. Thus, the edges incident to v in that part of the path can be replaced with edges from $C(v)$. It guarantees that Γ_{II} is a 2-cell embedding of a simple digraph such that each face is uniquely determined by a certain vertex v of type I and the boundary of this face is the directed cycle $C(v)$. So, this 2-cell embedding is closed. The digraph Γ_{II} is Eulerian by Lemma 1 and it is clear that $\Gamma = \mathrm{T}(\Gamma_{II})$.

We make the following observation, which will be used in the proof of the second part of the theorem. By (1) and (2) each zigzag of Γ which contains an edge of type II is homogeneous. The number of edges of type I is twice the number of edges of type II, so there is no zigzag passing through edges of type I only (since every edge is passed twice by a unique zigzag from τ or it is passed once by precisely two distinct zigzags from τ). Thus, if (1) and (2) are fulfilled, then all zigzags of Γ are homogeneous.

(II). Assume that Γ' is a closed 2-cell embedding of a simple Eulerian digraph such that edges of every face form a directed cycle.

Consider any face F of Γ' and let all edges of F form the directed cycle $e_1, \dots, e_n$. For every $i \in \{1, \dots, n\}$, we define $j(i) = i + 2 \pmod{n}$ and denote by e'_i and e''_i the edges containing the vertex v_F in $\mathrm{T}(\Gamma')$ such that each of these two edges has a common vertex with the edges e_i and $e_{j(i)}$ (respectively). We take the zigzag of $\mathrm{T}(\Gamma')$ containing the sequence $e_i, e'_i, e''_i, e_{j(i)}$. This zigzag goes through e_i and $e_{j(i)}$ in the directions of these edges. This is also true for every edge of Γ' occurring in this zigzag. Such a zigzag can be found for any pair consisting of a face of Γ' and one of edges of this face. The family of all such zigzags forms a z-orientation of $\mathrm{T}(\Gamma')$. All edges of Γ' are of type II and every v_F is a vertex of type I for this z-orientation. This implies that $\mathrm{T}(\Gamma')$ satisfies the conditions (1) and (2) which gives the claim. $\square$

The second part of Proposition 5 will be used to prove the implication (3) $\Rightarrow$ (1) in Theorem 6.

5.4 Structure of triangulations with faces of type I

Z-homogeneous triangulations form a subclass of z-oriented triangulations whose all faces are of type I. The main result of this subsection (Theorem 7) provides a topological description of such triangulations. One of its consequences is the implication (3) $\Rightarrow$ (1) in Theorem 6.

Recall that (Γ, τ) is a z-oriented triangulation of a surface S with all faces are of type I. The subgraph of Γ consisting of all vertices and all edges of type II will be denoted by Γ_{II}, as before. In the previous subsection we proved that if (Γ, τ) is z-homogeneous, then connected components of $S \setminus \Gamma_{II}$ are homeomorphic to open 2-dimensional discs. The following theorem describes connected components of $S \setminus \Gamma_{II}$ when Γ is not necessarily z-homogeneous.

Theorem 7. *The following assertions are fullfiled:*

(1) *Every connected components of $S \setminus \Gamma_{II}$ is homeomorphic to an open 2-dimensional disc or an open Möbius strip or an open cylinder.*

(2) *A connected component of $S \setminus \Gamma_{II}$ contains a vertex of type I if and only if it is an open 2-dimensional disc; such a vertex of type I is unique.*

Proof. Let F_1 be a face and let e_0 and e_1 be two distinct edges of type I belonging to F_1. Denote by F_2 the face which is distinct from F_1 and contains e_1 (there is precisely one such face). We write e_2 for the other edge of type I from F_2. By continuing this procedure, we get a sequence of edges $\{e_i\}_{i \in \mathbb{N} \cup \{0\}}$ and a sequence of faces $\{F_i\}_{i \in \mathbb{N}}$ such that e_{i-1} is the common edge of F_{i-1} and F_i for every $i \in \mathbb{N}$. The faces in each pair F_{i-1}, F_i can be adjacent in one of the two ways, as in Fig. 14. In the first case (Fig. 14 (a)), their edges of type II have a common vertex and in the second case (Fig. 14 (b)), the edges of type II are disjoint.

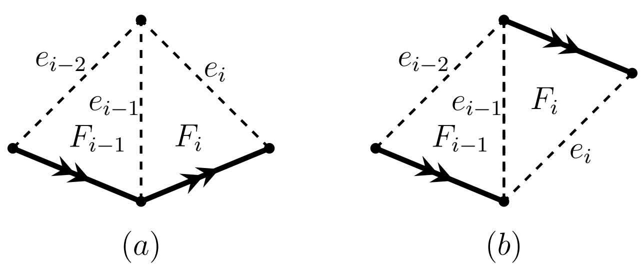

Figure 14

We take the smallest natural number n such that $e_n = e_0$ (the number of edges of Γ is finite, so we can find such a number). Then, the sequence of edges and the sequence of faces can be considered as cyclic sequences $\{e_i\}_{i=1}^n$ and $\{F_i\}_{i=1}^n$, respectively. The union of all faces from $\{F_i\}_{i=1}^n$ is said to be a *component* of (Γ, τ) and denoted by $\mathcal{F}$, i.e.

$$\mathcal{F} = \bigcup_{i=1}^{n} F_i.$$

The boundary of $\mathcal{F}$ consists of edges of type II from faces F_i, but not necessarily all of them are contained in the boundary.

We write e_i^{II} for the edge of type II from F_i. Let $T_1, T_2, \ldots, T_n$ be n disjoint closed triangles. For any $i = 1, 2, \ldots, n$ there is a homeomorphism $h_i : F_i \to T_i$ which transfers every vertex and every edge of F_i to a vertex and an edge of T_i (respectively). Now, for any i, we identify the edges $h_i(e_i)$ and $h_{i+1}(e_i)$ such that for every vertex v of e_i the vertices $h_i(v)$ and $h_{i+1}(v)$ are identified. In this way, we obtain a 2-dimensional surface $\mathcal{T}$ with boundary. The boundary of $\mathcal{T}$ is the union of the images of all edges of type II, i.e.

$$\partial\mathcal{T} = \bigcup_{i=1}^{n} h_i(e_i^{II}).$$

On the other hand, it is possible for distinct i, j that the edges e_i^{II}, e_j^{II} have a common vertex, so $\mathcal{F}$ is not necessarily a surface. The interior of surface $\mathcal{T}$ is homeomorphic to one of the connected components of $S \setminus \Gamma_{II}$, thus, $\mathcal{F}$ can be obtained from $\mathcal{T}$ by gluing of some parts of the boundary.

Assume that we identified $h_i(e_i)$ and $h_{i+1}(e_i)$ for $i = 1, 2, \ldots, n-1$, but $h_1(e_0)$ and $h_n(e_n)$ from T_1 and T_n (respectively) are not identified yet. Our space is homeomorphic to a closed 2-dimensional disc. Its boundary contains $h_1(e_0), h_n(e_n)$ and if we glue them, then we get $\mathcal{T}$. Therefore, we have two possibilities:

- A union of $h_1(e_0)$ and $h_n(e_n)$ is connected and if we glue them, then we obtain $\mathcal{T}$ homeomorphic to a closed 2-dimensional disc (Fig. 15 (1)).

- The edges $h_1(e_0)$ and $h_n(e_n)$ are disjoint and if we glue them, then we obtain $\mathcal{T}$ homeomorphic to a closed Möbius strip (Fig. 15 (2)) or a closed cylinder (Fig. 15 (3)).

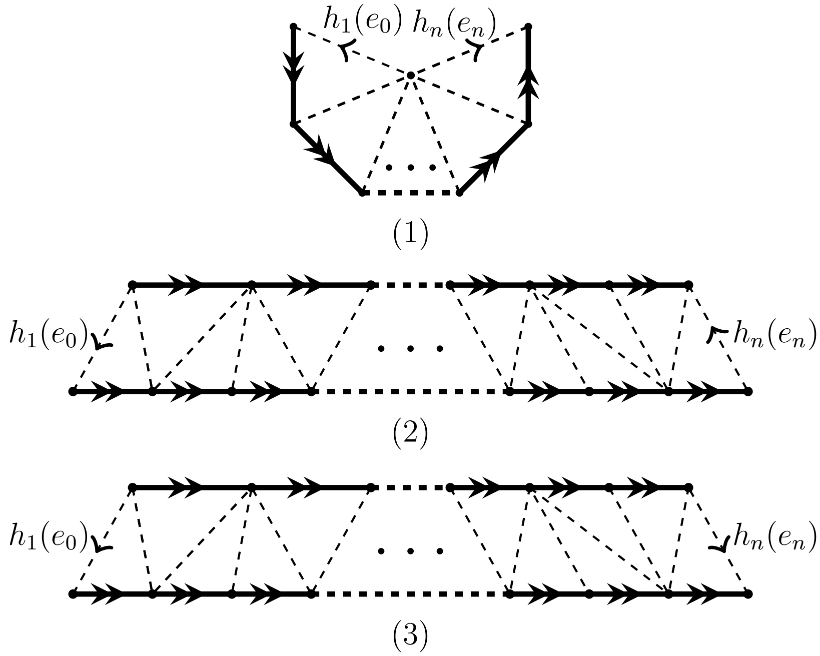

Figure 15

Denote by v_i the vertex of T_i which corresponds to the vertex of F_i not contained in the edge e_i^{II}. In the first case, all $h_i(v_i)$ are the same vertex in $\mathcal{T}$ and this vertex is the common vertex of the images of edges of type I. So, this vertex corresponds to the vertex of type I from $\mathcal{F}$, see Fig. 15 (1). In the remaining two cases, all vertices $h_i(v_i)$ belong to the boundary of $\mathcal{T}$ and correspond to certain vertices of Γ_{II}, see Fig. 15 (2) and Fig. 15 (3). The proof is completed. □

Remark 3. If a connected component of $S \setminus \Gamma_{II}$ is homeomorphic to an open 2-dimensional disc, then the corresponding component of (Γ, τ) is homeomorphic to a closed 2-dimensional disc. Indeed, if some parts of the boundary of this component are identified, then the vertex of type I contained in the component and a certain vertex at the boundary are connected by a double edge. This is impossible, since Γ is a simple graph.

Proof of (3)⇒(1) in Theorem 6. Suppose that all connected components of $S \setminus \Gamma_{II}$ are discs. By Remark 3, Γ_{II} is a closed 2-cell embedding and Lemma 1 guarantees that it is an embedding of simple Eulerian digraph. The statement (2) from Theorem 7 shows that each disc contains a unique vertex of type I. Repeating arguments from the proof of this theorem we establish that the boundary of each disc is an oriented cycle. Since $\Gamma = \mathrm{T}(\Gamma_{II})$, the required statement follows from the second part of Proposition 5. □

5.5 Examples

We present three examples of z-oriented triangulations with all faces of type I and consider the corresponding connected components of $S \setminus \Gamma_{II}$. These examples show that each type of components (an open disc, an open Möbius strip and an open cylinder) mentioned in Theorem 7 is realized.

Example 17. Let Γ be a z-oriented triangulation of a torus $\mathbb{T}$ presented in Fig. 16 (there is a z-orientation such that types of edges are as below). Thus, all faces are of type I. The subgraph Γ_{II} consists of two connected components which are 6-cycles. There are two connected components of $\mathbb{T} \setminus \Gamma_{II}$ and each of them is homeomorphic to an open cylinder.

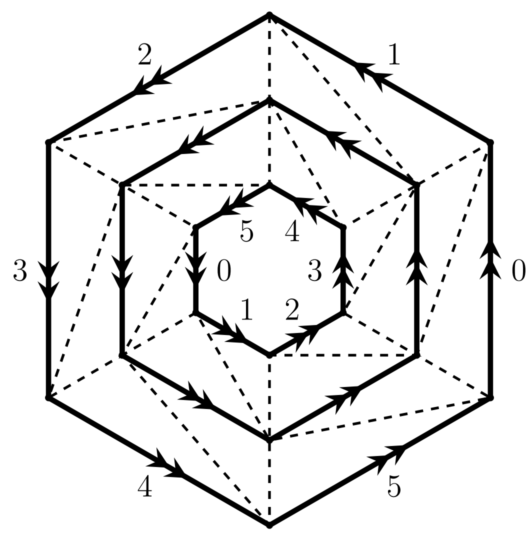

Figure 16

Example 18. Let $n \in \mathbb{N}$. Consider a triangulation Γ of a real projective plane $\mathbb{P}$ obtained by gluing of boundaries of a Möbius strip and a closed 2-dimensional disc, see Fig. 17. We take a z-orientation of Γ such that types of edges are as in Fig. 17. Thus, all faces are of type I and the subgraph Γ_{II} is a directed $2n$-cycle. Then $\mathbb{P}$ has two connected components: one of them is homeomorphic to an open 2-dimensional disc and the remaining is homeomorphic to an open Möbius strip.

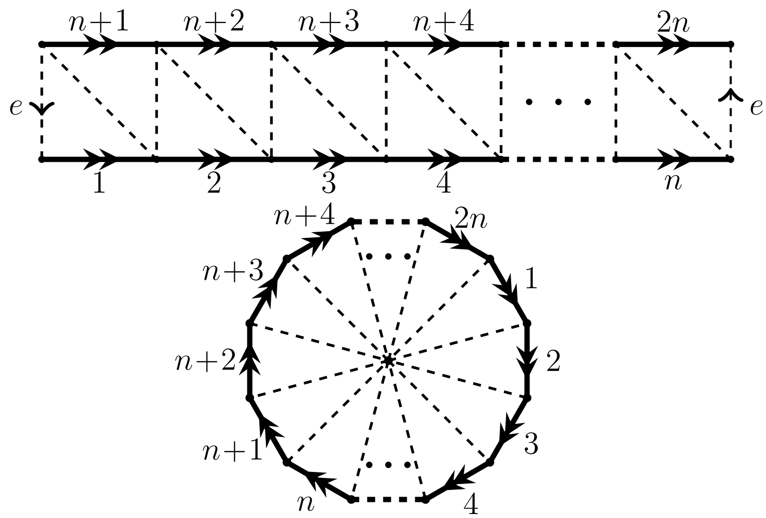

Figure 17

Example 19. Let Γ be a triangulation of a sphere presented in Fig. 18. This triangulation is obtained by the gluing of the two discs whose boundaries are cycles $e_1, e_2, \dots, e_6$. Let τ be a z-orientation such that types of all edges are as below. Thus all faces are of type I. Then $\mathbb{S} \setminus \Gamma_{II}$ consists of four connected components. Three of these connected components are homeomorphic to an open 2-dimensional disc and the remaining to an open cylinder. The components of (Γ, τ) corresponding to the open discs are closed 2-dimensional discs. The component of (Γ, τ) corresponding to the open cylinder is homeomorphic to a closed cylinder with two points from one of the connected components of its boundary identified.

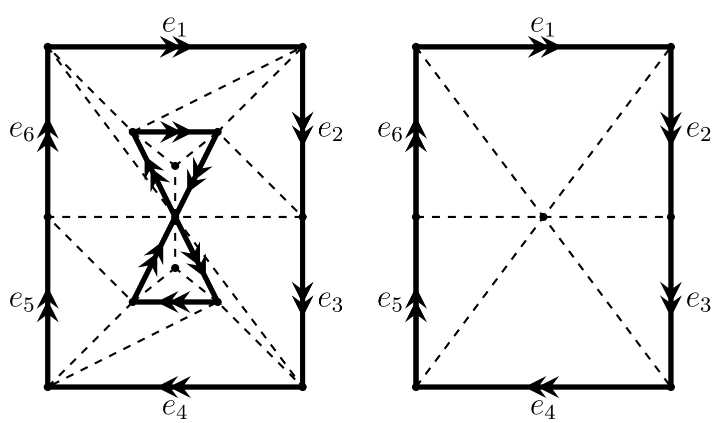

Figure 18

6 Z-knotted and z-homogeneous triangulations

In the previous two sections we investigated z-knotted triangulations and z-homogeneous triangulations of surfaces. By Theorem 4, for any triangulation there is a z-knotted shredding. There is a one-to-one correspondence between z-homogeneous triangulations and embeddings of Eulerian digraphs (Theorem 6). In the present section, following [30], we describe an algorithm which transforms any z-homogeneous triangulation into a z-homogeneous and z-knotted triangulation and does not change the surface type.

6.1 Main result

We describe a special type of the connected sum of triangulated surfaces different from the sum used in Section 4.

Let Γ be a triangulation of a surface S. Suppose that τ is a z-orientation of Γ such that (Γ, τ) is z-homogeneous.

Let P be a path in Γ_{II} formed by oriented edges $e_1 = v_1v_2$ and $e_2 = v_2v_3$, where v_1, v_2, v_3 are certain vertices of Γ_{II} (see Fig. 19). Such pairs are said to be *special*. For $i = 1, 2$, the both faces whose one of edges is e_i will be denoted by F_i^+ and F_i^-. For each $\delta \in \{+, -\}$, we suppose that the faces F_1^δ, F_2^δ are on the same side of the path P. We split up both edges e_i in two oriented edges e_i^+ and e_i^- whose directions are induced by the direction of e_i and such that e_i^δ is an edge of the face F_i^δ. Also, we split up the vertex v_2 in two vertices v_2^+ and v_2^- such that $e_1^\delta = v_1v_2^\delta$ and $e_2^\delta = v_2^\delta v_3$. As a result, we get a new graph $N_P(\Gamma)$ embedded in S. The set of faces of $N_P(\Gamma)$ is the set of faces of Γ extended by a new 4-gonal face F_P, see Fig. 19.

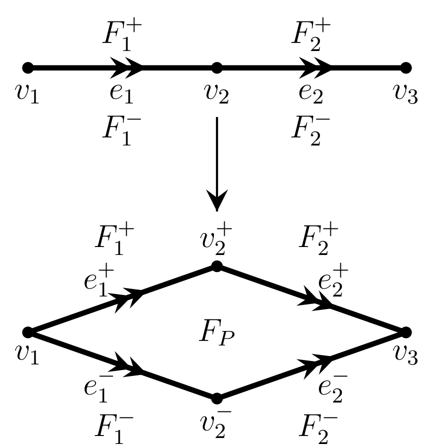

Figure 19

We take another z-homogeneous triangulation (Γ', τ') of a surface S'. Denote by Γ'_{II} the subgraph of Γ' whose the set of edges and the set of vertices consist of all edges of type II and all vertices of type II, respectively. Let P' be a special pair in Γ'_{II}. We construct the graph $N_{P'}(\Gamma')$ for this special pair. Moreover, we assume the following:

($*$) for at least one of the pairs P, P', the endpoints of this pair are not connected by an edge in Γ or Γ', respectively.

In other words, there is no cycle of length 3 containing edges of at least one of the pairs P, P'.

The orientations of edges of F_P and $F_{P'}$ are determined by the z-orientations τ and τ', respectively. The sets of all such oriented edges of F_P and $F_{P'}$ will be denoted by $\omega(F_P)$ and $\omega(F_{P'})$ (respectively). The notion of special homeomorphism defined in Section 2 can be generalized on homeomorphisms of 4-gonal faces preserving the orientations of edges, i.e. a homeomorphism $g : \partial F_P \to \partial F_{P'}$ is said to be *special* if it transfers the vertices of F_P to the vertices of $F_{P'}$ and the edges from $\omega(F_P)$ to the edges from $\omega(F_{P'})$. We glue $N_P(\Gamma)$ and $N_{P'}(\Gamma')$ by the special homeomorphism g and obtain a triangulation of $S\#S'$ which will be denoted by $\Gamma\#_g\Gamma'$. Since we assume ($*$), the triangulation $\Gamma\#_g\Gamma'$ does not contain multiple edges. The special homeomorphism g preserves the orientations of edges, so, there exists a z-orientation of $\Gamma\#_g\Gamma'$ such that the type of every edge is the same as in (Γ, τ) or (Γ', τ'). Thus, $\Gamma\#_g\Gamma'$ is z-homogeneous for this z-orientation.

Let $P_1, \dots, P_n$ be special pairs in (Γ, τ) and let $P'_1, \dots, P'_n$ be special pairs in z-homogeneous triangulations $(\Gamma_1, \tau_1), \dots, (\Gamma_n, \tau_n)$ of a sphere $\mathbb{S}$. We take special homeomorphisms $g_i : \partial F_{P_i} \to \partial F_{P'_i}$ for $i \in \{1, \dots, n\}$. We apply the above operation of connected sum several times and obtain a triangulation of S

$$(((\Gamma\#_{g_1}\Gamma_1)\#_{g_2}\Gamma_2)\dots)\#_{g_n}\Gamma_n \tag{2}$$

which is z-homogeneous for a certain z-orientation.

Theorem 8 ([30])**.** *For any z-homogeneous triangulation (Γ, τ) with $|\tau| > 1$ there exist z-homogeneous triangulations $(\Gamma_1, \tau_1), \dots, (\Gamma_n, \tau_n)$ of a sphere $\mathbb{S}$ such that $n \le |\tau| - 1$ and the z-homogeneous connected sum (2) is z-knotted for some special pairs P_i in Γ, P'_i in Γ_i and special homeomorphisms $g_i : \partial F_{P_i} \to \partial F_{P'_i}$ with $i \in \{1, \dots, n\}$.*

6.2 Z-monodromy of special pairs

The z-monodromy of faces was investigated in Section 3. Now, we consider a similar concept for special pairs in z-homogeneous triangulations.

Let (Γ,τ) be a z-homogeneous triangulation and let P be a special pair in Γ_{II}. Consider the set of all oriented edges of 4-gonal face F_P in $N_P(\Gamma)$:

$$\Omega(F_P)=\{e_1^+,e_2^+,e_1^-,e_2^-,-e_1^+,-e_2^+,-e_1^-,-e_2^-\}.$$

The edges e_i^δ and $-e_i^\delta$ comes from the same non-oriented edge, but they are considered with opposite orientations. Denote by D_{F_P} the permutation on $\Omega(F_P)$ transferring ab to bc for any three consecutive vertices a,b,c belonging to F_P. Thus, D_{F_P} is the composition of two distinct commuting 4-cycles

$$D_{F_P}=(e_1^+,e_2^+,-e_2^-,-e_1^-)(e_1^-,e_2^-,-e_2^+,-e_1^+).$$

Each of these cycles gives one of the orientations on the boundary of F_P.

For any pair $e_0,e\in\Omega(F_P)$ such that $D_{F_P}(e_0)=e$ there exists the zigzag containing the sequence e_0,e. Let the first element of $\Omega(F_P)$ contained in this zigzag after e be denoted by $M_{F_P}(e)$. The proof that $M_{F_P}:\Omega(F_P)\to\Omega(F_P)$ is bijective is similar to the proofs of the statements (1) and (2) from Lemma 4. The restriction of M_{F_P} to the set

$$\omega(F_P)=\{e_1^+,e_2^+,e_1^-,e_2^-\}$$

is said to be the *z-monodromy* of P and denoted by M_P.

We show that M_P is a permutation on $\omega(F_P)$. For each $e_i^\delta\in\omega(F_P)$ we take the following part of a zigzag in $N_P(\Gamma)$

$$[e_i^\delta,D_{F_i^\delta}(e_i^\delta),\dots,M_P(e_i^\delta)].$$

It corresponds to a part of a zigzag from Γ contained in τ which is

$$[e_i,D_{F_i^\delta}(e_i),\dots,e_j],$$

where $j\in\{1,2\}$. Thus, $M_P(e_i^\delta)=e_j^\gamma$, where $\gamma\in\{+,-\}$, and $M_P(e_i^\delta)\in\omega(F_P)$. Therefore, M_P can be identified with a certain permutation from the symmetric group S_4.

For simplicity of notation, from this moment we write $1,2$ instead of e_1^+,e_2^+ and $3,4$ instead of e_1^-,e_2^- (respectively). The symmetric group S_4 consists of 24 permutations, so there are 24 possibilities for M_P. Their realization will be investigated later. We split all these possibilities into the following 13 classes:

(K_0) id,

(K_1) (1234),

(K_2) $(13)(24)$,

(K_3) (1432),

(K_4) $(14)(23)$,

(K_5) $(12)(34)$,

(K_6) $(24),(13)$,

(K_7) $(34),(12)$,

(K_8) $(23),(14)$,

(K_9) $(1324),(1423)$,

(K_{10}) $(1243),(1342)$,

(K_{11}) $(234),(123),(124),(134)$,

(K_{12}) $(243),(132),(142),(143)$.

We explain why permutations from S_4 were decomposed as above. Define the following two commuting involutions:

$$s=(13)(24), \qquad t=(12)(34).$$

Two permutations p_1, p_2 are elements of the same class K_i if one of the following two possibilities is realized:

$$p_2=sp_1s \quad \text{or} \quad p_2=up_1^{-1}u,$$

where $u\in\{t,st\}$, see Tab. 1 in Appendix (Subsection 6.5). The permutation s corresponds to the interchanging of $+$ and $-$ for the edges e_i^{δ}. The changing of the z-orientation τ to τ^{-1} replaces a permutation p corresponding to the z-monodromy by $tp^{-1}t$.

6.3 Examples

In this subsection we present examples which show that the z-monodromy from each of the classes mentioned in the previous subsection is realized.

Example 20. Let Γ be the bipyramid BP_n with the z-orientation such that BP_n is z-homogeneous, see Example 15. Denote by P a special pair in BP_n. In contrast to our previous notation, we denote the consecutive vertices of the base by $v_1, \dots, v_n$ to avoid notational conflicts with elements of $\omega(F_P)$. Note that any special pair of BP_n can be transferred to any another special pair by an automorphism, so, for all special pairs in BP_n the z-monodromies belong to the same class. Thus, we can assume that the directed edges of P are v_1v_2 and v_2v_3. In z-monodromies, these edges will be represented by 1 and 2, respectively, if they are contained in the faces with the vertex a in $N_P(BP_n)$. If these edges are contained in the faces with the vertex b in $N_P(BP_n)$, then we write for them 3 and 4 in z-monodromies (respectively).

The case when $n = 2k + 1$ and k is odd. The zigzag passes through edges of P as follows

$$\dots, av_1, v_1v_2, v_2b, \dots, av_2, v_2v_3, v_3b, \dots, bv_1, v_1v_2, v_2a, \dots, bv_2, v_2v_3, v_3a, \dots$$

and the corresponding union of parts of zigzags in $N_P(BP_n)$ is

$$\dots, 1, 3, \dots, 2, 4, \dots, 3, 1, \dots, 4, 2, \dots$$

and the z-monodromy is

$$M_P = (1432)$$

from the class K_3.

The case when $n = 2k + 1$ and k is even. The zigzag passing through the edges of P in BP_n is

$$\dots, av_1, v_1v_2, v_2b, \dots, bv_2, v_2v_3, v_3a, \dots, bv_1, v_1v_2, v_2a, \dots, av_2, v_2v_3, v_3b, \dots$$

and the corresponding union of parts of zigzags in $N_P(BP_n)$ is

$$\dots, 1, 3, \dots, 4, 2, \dots, 3, 1, \dots, 2, 4, \dots$$

which means that the z-monodromy is

$$M_P = (1234)$$

from the class K_1.

The case when $n = 2k$ and k is odd. The edges of P occur in zigzags of BP_n as follows:

$$\dots, av_1, v_1v_2, v_2b, \dots, bv_1, v_1v_2, v_2a, \dots$$

and

$$\ldots, av_2, v_2v_3, v_3b, \ldots, bv_2, v_2v_3, v_3a, \ldots;$$

the corresponding unions of parts of zigzags in $N_P(BP_n)$ are

$$\ldots, 1, 3, \ldots, 3, 1, \ldots \quad \text{and} \quad \ldots, 2, 4, \ldots, 4, 2, \ldots$$

which implies that M_P is the identity (class K_0).

The case when $n = 2k$ and k is even. The four zigzags of BP_n pass through the edges of P as follows

$$\ldots, av_1, v_1v_2, v_2b, \ldots,$$

$$\ldots, bv_1, v_1v_2, v_2a, \ldots,$$

$$\ldots, av_2, v_2v_3, v_3b, \ldots,$$

$$\ldots, bv_2, v_2v_3, v_3a, \ldots$$

and the corresponding parts of zigzags in $N_P(BP_n)$ are

$$\ldots, 1, 3, \ldots, \quad \ldots, 3, 1, \ldots, \quad \ldots, 2, 4, \ldots, \quad \ldots, 4, 2, \ldots$$

and the z-monodromy

$$M_P = (13)(24)$$

is from the class K_2.

We give a construction of an infinite series of z-homogeneous triangulations of $\mathbb{S}$ that are not bipyramids. Let Γ' be a graph embedded in $\mathbb{S}$ which consists of two vertices a, b and four paths P_1, P_2, P_3, P_4. We assume that at most one of these paths is an edge and any two of these paths intersect precisely in a and b, see Fig. 20. Thus, Γ' is a connected finite simple graph and $\mathbb{S}$ is decomposed into four discs whose boundaries are $P_1 \cup P_2$, $P_2 \cup P_3$, $P_3 \cup P_4$ and $P_1 \cup P_4$. These discs are faces of our embedding. We apply the operation described at the beginning of Subsection 5.3 to the embedding of Γ' in $\mathbb{S}$ and obtain a triangulation $\mathrm{T}(\Gamma')$ of $\mathbb{S}$. If p_i is the number of edges in the path P_i, then the triangulation $\mathrm{T}(\Gamma')$ will be denoted by Γ_{p_1,p_2,p_3,p_4}. For example, see Fig. 20 for $\Gamma_{2,3,3,3}$. Note that any cyclic permutation of p_1, p_2, p_3, p_4 does not change the triangulation. We appoint an orientation on each edge of all P_i such that the boundary of every disc is formed by a path from a to b and a path from b to a. By Proposition 5 the triangulation Γ_{p_1,p_2,p_3,p_4} is z-homogeneous for a certain z-orientation such that the subgraph Γ' consists of all edges of type II and their vertices.

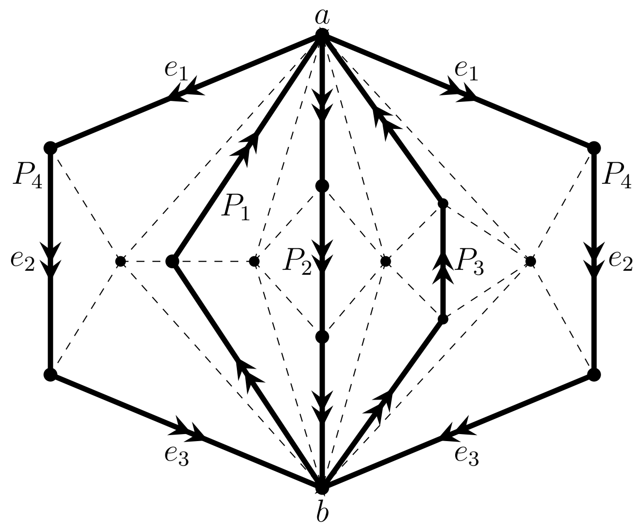

Figure 20

Example 21. Consider the z-homogeneous triangulation $\Gamma_{2,3,4,5}$ (with the corresponding z-orientation) containing the paths

$$P_1 = \{a, v_0, b\},$$

$$P_2 = \{b, v_1, v_2, a\},$$
$$P_3 = \{a, v_3, v_4, v_5, b\},$$
$$P_4 = \{b, v_6, v_7, v_8, v_9, a\}.$$

Let v_{ij} be the vertex in the interior of a disc whose boundary is $P_i \cup P_j$ (for $i, j \in \{1, 2, 3, 4\}$ and $i < j$). The z-orientation of $\Gamma_{2,3,4,5}$ contains precisely two zigzags presented below as cyclic sequences of vertices:

$$v_{14}, a, v_0, v_{12}, b, v_1, v_{23}, v_2, a, v_{12}, v_0, b, v_{14}, v_6, v_7, v_{34}, v_8, v_9$$

and

$$v_{14}, v_0, b, v_{12}, v_1, v_2, v_{23}, a, v_3, v_{34}, v_4, v_5, v_{23}, b, v_1, v_{12}, v_2, a, v_{23}, v_3, v_4,$$

$$v_{34}, v_5, b, v_{23}, v_1, v_2, v_{12}, a, v_0, v_{14}, b, v_6, v_{34}, v_7, v_8, v_{14}, v_9, a, v_{34}, v_3, v_4,$$

$$v_{23}, v_5, b, v_{34}, v_6, v_7, v_{14}, v_8, v_9, v_{34}, a, v_3, v_{23}, v_4, v_5, v_{34}, b, v_6, v_{14}, v_7, v_8, v_{34}, v_9, a.$$

(1). If P is the special pair v_0b, bv_1, then the zigzags which pass through the edges of P are

$$\ldots, v_{12}b, bv_1, v_1v_{23}, \ldots, v_{12}v_0, v_0b, bv_{14}, \ldots$$

and

$$\ldots, v_{14}v_0, v_0b, bv_{12}, \ldots, v_{23}b, bv_1, v_1v_{12}, \ldots$$

We denote by $1, 2$ the edges from $N_P(\Gamma_{2,3,4,5})$ which correspond to v_0b, bv_1 (respectively) and belong to the faces containing v_{12}. We write $3, 4$ for the edges from $N_P(\Gamma_{2,3,4,5})$ corresponding to v_0b, bv_1 which belong to the faces whose vertices are v_{14} and v_{23}, respectively. The above zigzags gives the following unions of parts of zigzags in $N_P(\Gamma_{2,3,4,5})$

$$\ldots, 2, 4, \ldots, 1, 3, \ldots \quad \text{and} \quad \ldots, 3, 1, \ldots, 4, 2, \ldots$$

Thus, the z-monodromy is

$$M_P = (14)(23)$$

and it belongs to the class K_4.

(2). If P is the special pair av_0, v_0b, then the zigzags

$$\ldots, v_{14}a, av_0, v_0v_{12}, \ldots, v_{12}v_0, v_0b, bv_{14}, \ldots$$

and

$$\ldots, v_{14}v_0, v_0b, bv_{12}, \ldots, v_{12}a, av_0, v_0v_{14}, \ldots$$

pass through the edges of P. We denote by $1, 2$ the edges from $N_P(\Gamma_{2,3,4,5})$ corresponding to av_0, v_0b (respectively) which are in the faces containing v_{12}. We write $3, 4$ for the edges related to av_0, v_0b which are in the faces containing v_{14}, respectively. The above zigzags gives the following unions of parts of zigzags in $N_P(\Gamma_{2,3,4,5})$

$$\ldots, 3, 1, \ldots, 2, 4, \ldots \quad \text{and} \quad \ldots, 4, 2, \ldots, 1, 3, \ldots$$

Thus, the z-monodromy is

$$M_P = (12)(34)$$

and it belongs to the class K_5.

(3). If P is the special pair v_3v_4, v_4v_5, then the zigzag

$$\ldots, v_{34}v_4, v_4v_5, v_5v_{23}, \ldots, v_{23}v_3, v_3v_4, v_4v_{34}, \ldots, v_{34}v_3, v_3v_4, v_4v_{23}, \ldots, v_{23}v_4, v_4v_5, v_5v_{34}, \ldots$$

passes through the edges of P. We denote by $1, 2$ the edges from $N_P(\Gamma_{2,3,4,5})$ which correspond to v_3v_4, v_4v_5 (respectively) and belong to the faces containing v_{23}. We write $3, 4$ for the edges from $N_P(\Gamma_{2,3,4,5})$ associated to v_3v_4, v_4v_5 which belong to the faces whose vertex is v_{34}, respectively. The corresponding union of parts of zigzags in $N_P(\Gamma_{2,3,4,5})$ is

$$\ldots, 4, 2, \ldots, 1, 3, \ldots, 3, 1, \ldots, 2, 4, \ldots$$

and the z-monodromy

$$M_P = (12)$$

belongs to the class K_7.

(4). If P is the special pair av_3, v_3v_4, then the zigzag

$$\ldots, v_{23}a, av_3, v_3v_{34}, \ldots, v_{23}v_3, v_3v_4, v_4v_{34}, \ldots, v_{34}v_3, v_3v_4, v_4v_{23}, \ldots, v_{34}a, av_3, v_3v_{23}, \ldots$$

passes through the edges of P. In $N_P(\Gamma_{2,3,4,5})$, we denote by $1, 2$ the edges corresponding to av_3, v_3v_4 (respectively) which are in the faces containing v_{23}. We write $3, 4$ for the edges from $N_P(\Gamma_{2,3,4,5})$ corresponding to av_3, v_3v_4 which are in the faces containing v_{34}, respectively. The associated union of parts of zigzags in $N_P(\Gamma_{2,3,4,5})$ is

$$\ldots, 1, 3, \ldots, 2, 4, \ldots, 4, 2, \ldots, 3, 1, \ldots$$

and the z-monodromy

$$M_P = (23)$$

is from the class K_8.

(5). If P is the special pair bv_6, v_6v_7, then the zigzags

$$\ldots, v_{14}v_6, v_6v_7, v_7v_{34} \ldots$$

and

$$\ldots, v_{14}b, bv_6, v_6v_{34}, \ldots, v_{34}v_6, v_6v_7, v_7v_{14}, \ldots, v_{34}b, bv_6, v_6v_{14}, \ldots$$

pass through the edges of P. We denote by $1, 2$ the edges from $N_P(\Gamma_{2,3,4,5})$ which correspond to bv_6, v_6v_7 (respectively) and belong to the faces containing v_{14}. We write $3, 4$ for the edges from $N_P(\Gamma_{2,3,4,5})$ related to bv_6, v_6v_7 which belong to the faces whose vertex is v_{34}, respectively. The corresponding unions of parts of zigzags in $N_P(\Gamma_{2,3,4,5})$ are

$$\ldots, 2, 4, \ldots \quad \text{and} \quad \ldots, 1, 3, \ldots, 4, 2, \ldots, 3, 1, \ldots$$

which implies that the z-monodromy

$$M_P = (234)$$

is from the class K_{11}.

(6). If P is the special pair bv_1, v_1v_2, then the zigzags

$$\ldots, v_{12}b, bv_1, v_1v_{23}, \ldots$$

and

$$\ldots, v_{12}v_1, v_1v_2, v_2v_{23}, \ldots, v_{23}b, bv_1, v_1v_{12}, \ldots, v_{23}v_1, v_1v_2, v_2v_{12}, \ldots$$

pass through the edges of P. In $N_P(\Gamma_{2,3,4,5})$, we denote by $1,2$ the edges corresponding to bv_1, v_1v_2 (respectively) which are in the faces containing v_{12}. We write $3,4$ for the edges from $N_P(\Gamma_{2,3,4,5})$ corresponding to bv_1, v_1v_2 which are in the faces containing v_{23}, respectively. The above zigzags gives the following unions of parts of zigzags in $N_P(\Gamma_{2,3,4,5})$

$$\ldots, 1, 3, \ldots \quad \text{and} \quad \ldots, 2, 4, \ldots, 3, 1, \ldots, 4, 2, \ldots$$

and this implies that the z-monodromy

$$M_P = (143)$$

belongs to the class K_{12}.

Example 22. Consider the z-homogeneous triangulation $\Gamma_{2,4,3,4}$ (with the corresponding z-orientation) with the following four paths

$$P_1 = \{a, v_0, b\},$$

$$P_2 = \{b, v_1, v_2, v_3, a\},$$
$$P_3 = \{a, v_4, v_5, b\},$$
$$P_4 = \{b, v_6, v_7, v_8, a\}.$$

As previous, we denote by v_{ij} the vertex in the interior of a disc whose boundary is $P_i \cup P_j$ (for $i, j \in \{1,2,3,4\}$ and $i < j$). The z-orientation of $\Gamma_{2,4,3,4}$ consists of the following three zigzags:

$$v_{14}, a, v_0, v_{12}, b, v_1, v_{23}, v_2, v_3, v_{12}, a, v_0, v_{14}, b, v_6, v_{34}, v_7, v_8$$

and

$$v_{14}, v_0, b, v_{12}, v_1, v_2, v_{23}, v_3, a, v_{12}, v_0, b, v_{14}, v_6, v_7, v_{34}, v_8, a$$

and

$$v_{23}, b, v_1, v_{12}, v_2, v_3, v_{23}, a, v_4, v_{34}, v_5, b, v_{23}, v_1, v_2, v_{12}, v_3, a, v_{23}, v_4, v_5,$$
$$v_{34}, b, v_6, v_{14}, v_7, v_8, v_{34}, a, v_4, v_{23}, v_5, b, v_{34}, v_6, v_7, v_{14}, v_8, a, v_{34}, v_4, v_5.$$

(1). If P is the special pair v_0b, bv_1, then the zigzags

$$\ldots, v_{12}b, bv_1, v_1v_{23}, \ldots$$

and

$$\ldots, v_{14}v_0, v_0b, bv_{12}, \ldots, v_{12}v_0, v_0b, bv_{14}, \ldots$$

and

$$\ldots, v_{23}b, bv_1, v_1v_{12}, \ldots$$

pass through the edges of P. We denote by $1, 2$ the edges from $N_P(\Gamma_{2,4,3,4})$ which correspond to v_0b, bv_1 (respectively) and belong to the faces containing v_{12}. We write $3, 4$ for the edges from $N_P(\Gamma_{2,4,3,4})$ corresponding to v_0b, bv_1 which are in the faces containing v_{14} and v_{23}, respectively. The associated unions of parts of zigzags in $N_P(\Gamma_{2,4,3,4})$ are

$$\ldots, 2, 4, \ldots, \qquad \ldots, 3, 1, \ldots, 1, 3, \ldots, \qquad \ldots, 4, 2, \ldots$$

and the z-monodromy

$$M_P = (24)$$

is from the class K_6.

(2). If P is the special pair bv_1, v_1v_2 then the zigzags

$$\ldots, v_{12}b, bv_1, v_1v_{23}, \ldots$$

and

$$\ldots, v_{12}v_1, v_1v_2, v_2v_{23}, \ldots$$

and

$$\ldots, v_{23}b, bv_1, v_1v_{12}, \ldots, v_{23}v_1, v_1v_2, v_2v_{12}, \ldots$$

pass through the edges of P. In $N_P(\Gamma_{2,4,3,4})$, we denote by $1, 2$ the edges corresponding to bv_1, v_1v_2 (respectively) which are in the faces containing v_{12}. We write $3, 4$ for the edges related to bv_1, v_1v_2 which are in the faces containing v_{23}, respectively. The corresponding unions of parts of zigzags in $N_P(\Gamma_{2,4,3,4})$ are

$$\ldots, 1, 3, \ldots, \qquad \ldots, 2, 4, \ldots, \qquad \ldots, 3, 1, \ldots, 4, 2, \ldots$$

which implies that the z-monodromy

$$M_P = (1423)$$

belongs to the class K_9.

(3). If P is the special pair v_1v_2, v_2v_3 then the zigzags

$$\ldots, v_{23}v_2, v_2v_3, v_3v_{12}, \ldots$$

and

$$\ldots, v_{12}v_1, v_1v_2, v_2v_{23}, \ldots$$

and

$$\ldots, v_{12}v_2, v_2v_3, v_3v_{23}, \ldots, v_{23}v_1, v_1v_2, v_2v_{12}, \ldots$$

pass through the edges of P. We denote by $1, 2$ the edges from $N_P(\Gamma_{2,4,3,4})$ associated to v_1v_2, v_2v_3 (respectively) which belong to the faces containing v_{12}. We write $3, 4$ for the edges from $N_P(\Gamma_{2,4,3,4})$ related to v_1v_2, v_2v_3 which are in the faces containing v_{23}, respectively. The related unions of parts of zigzags in $N_P(\Gamma_{2,4,3,4})$ are

$$\ldots, 4, 2, \ldots, \qquad \ldots, 1, 3, \ldots, \qquad \ldots, 2, 4, \ldots, 3, 1, \ldots$$

and the z-monodromy

$$M_P = (1243)$$

is from the class K_{10}.

6.4 Proof of Theorem 8

Let Γ and Γ' be triangulations and suppose that there are z-orientations such that Γ and Γ' are z-homogeneous. Let P and P' be special pairs in Γ and Γ' (respectively). As in the previous subsection, the faces in $N_P(\Gamma)$ and $N_{P'}(\Gamma')$ obtained from P and P' are denoted by F_P and $F_{P'}$, respectively.

Lemma 9. *The number of zigzags from the z-orientation of Γ passing through the edges of P is equal to the number of cycles in the permutation sM_P where $s = (13)(24)$.*

Proof. Let $e \in \omega(F_P) = \{1, 2, 3, 4\}$ and let $[e, M_P(e)]$ be the part of a zigzag in $N_P(\Gamma)$ such that e is its the first element, $M_P(e)$ is its the last element and any other element from $\omega(F_P)$ is not contained in this path. Consider the cyclic sequence

$$[e, M_P(e)], [sM_P(e), M_PsM_P(e)], \ldots, [(sM_P)^{m-1}(e), M_P(sM_P)^{m-1}(e)], \tag{3}$$

where m is the smallest positive number such that $(sM_P)^m(e) = e$. Since the edges $e', se' \in \omega(F_P)$ correspond to the same edge of Γ, by identifying the last edge of every part of (3) with the first edge of the next part we get a zigzag in Γ. Thus, for every cycle in sM_P there is a zigzag which pass through the edges of P. Note that other cycle of sM_P defines a different zigzag. On the other hand, by the definition of z-monodromy, each zigzag of Γ passing through the edges of P induces a cycle in sM_P. Therefore, there is a one-to-one correspondence between zigzags passing through the edges of P and cycles in the permutation sM_P. □

Lemma 10. *If k is the number of zigzags from the z-orientation passing through the edges of P, then the following assertions are fulfilled:*

- $k = 1$ *if the z-monodromy of P is from the class K_i with $i \in \{1, 3, 7, 8\}$;*
- $k = 2$ *if the z-monodromy of P is from the class K_i with $i \in \{0, 4, 5, 11, 12\}$;*
- $k = 3$ *if the z-monodromy of P is from the class K_i with $i \in \{6, 9, 10\}$;*
- $k = 4$ *if the z-monodromy of P is from the class K_2.*

Proof. All permutations from the class K_i contain the same number of cycles since q is conjugate to p or p^{-1} for any two different permutation $p, q \in K_i$. We apply Lemma 9 to an element from every of the 13 classes (recall that each fixed point is a 1-cycle):

(K_0) $s(id) = s = (13)(24)$,

(K_1) $s(1234) = (1432)$,

(K_2) $s(13)(24) = s^2 = id$,

(K_3) $s(1432) = (1234)$,

(K_4) $s(14)(23) = (12)(34)$,

(K_5) $s(12)(34) = (14)(23)$,

(K_6) $s(24) = (13)$,

(K_7) $s(34) = (1324)$,

(K_8) $s(23) = (1342)$,

(K_9) $s(1324) = (34)$,

(K_{10}) $s(1243) = (14)$,

(K_{11}) $s(234) = (132)$,

(K_{12}) $s(243) = (134)$.

$\square$

We say that a special pair P is *essential* if each zigzag belonging to the z-orientation passes through the edges of P.

Example 23. We give examples of essential pairs:

(1) Any special pair in the bipyramid BP_n is essential (see Example 20) and the z-monodromies of such pairs are from the class K_i for $i \in \{0,1,2,3\}$.

(2) The following special pairs in the triangulation $\Gamma_{2,3,4,5}$ (see Example 21) are essential: v_0b, bv_1 (the class K_4), av_0, v_0b (the class K_5), bv_6, v_6v_7 (the class K_{11}), bv_1, v_1v_2 (the class K_{12}).

(3) The following special pairs in the triangulation $\Gamma_{2,4,3,4}$ (see Example 22) are essential: v_0b, bv_1 (the class K_6), bv_1, v_1v_2 (the class K_9), v_1v_2, v_2v_3 (the class K_{10}).

Lemma 11. *Let $g : \partial F_P \to \partial F_{P'}$ be a special homeomorphism. The number of zigzags (up to reversing) in $\Gamma \#_g \Gamma'$ passing through the edges of*

$$g(\omega(F_P)) = \omega(F_{P'})$$

is equal to the number of cycles in $g^{-1}M_{P'}gM_P$.

Proof. The proof is similar to the proofs of Lemma 5 and Lemma 9. We consider the following cyclic sequence

$$[e, M_P(e)], [gM_P(e), M_{P'}gM_P(e)], [g^{-1}M_{P'}gM_P(e), M_P(g^{-1}M_{P'}gM_P)(e)],$$

$$\dots$$

$$[gM_P(g^{-1}M_{P'}gM_P)^{m-1}(e), M_{P'}gM_P(g^{-1}M_{P'}gM_P)^{m-1}(e)],$$

where m is the smallest positive number such that $(g^{-1}M_{P'}gM_P)^m(e) = e$. By identifying the last edge of any part with the first edge of the next part we get a zigzag in $\Gamma \#_g \Gamma'$. Thus, there is a one-to-one correspondence between zigzags passing through edges $g(\omega(F_P)) = \omega(F_{P'})$ and cycles in the permutation $g^{-1}M_{P'}gM_P$. $\square$

We prove Theorem 8. Suppose that Γ is not z-knotted, i.e. the z-orientation of Γ consists of more than one zigzag. Then Γ contains a face F such that there are at least two zigzags in the z-orientation which pass through the edges of F. One of these zigzags pass through the edge of type II from F. We denote this edge by e. There are two possibilities:

- e occurs in two distinct zigzags;

• e occurs twice in one zigzag Z.

It is obvious that in the first case there is a special pair P whose edges are contained in more than one zigzag from the z-orientation. We show that the same is true in the second case. A zigzag from the z-orientation and distinct from Z passes through the edges of type I belonging to F. So, the next edge e' in this zigzag belongs to a face adjacent to F and it is easy to see that e' has a common vertex with e. Since this zigzag is homogeneous, e' is of type II. Thus, the edges e and e' form a special pair.

Since there are at least two zigzags from the z-orientation passing through the edges of P, then M_P belongs to K_i where $i \in \{0, 2, 4, 5, 6, 9, 10, 11, 12\}$. Recall that Γ' is a z-homogeneous triangulation containing a special pair P'. From this moment, we suppose that Γ' is a triangulation of $\mathbb{S}$ and P' is an essential pair. Using Lemma 11 we establish that if $g : \partial F_P \to \partial F_{P'}$ is a special homeomorphism such that $g^{-1}M_{P'}gM_P$ is a 4-cycle, then the number of zigzags in $\Gamma \#_g \Gamma'$ is less than the number of zigzags in Γ. Now, we show that there are suitable Γ' and P' in all nine possibilities of M_P.

As previous, let $\omega(F) = \{1, 2, 3, 4\}$. Suppose that $\omega(F') = \{1', 2', 3', 4'\}$. In the proof we will exploit the z-monodromies of essential pairs in triangulations of a sphere from Example 23. Each gluing will be made using the special homeomorphism $g : \partial F_P \to \partial F_{P'}$ transferring every $k \in \omega(F_P)$ to k'.

If there is more than one permutation in the class K_i, then for any two distinct $p, q \in K_i$ the permutation p can be obtained from q by the renumerations of the edges in the 4-gonal face F_P and the change of the z-orientation by the opposite. Thus, it is sufficient to find suitable Γ' and P' only for one permutation from such K_i.

(K_0). If $M_P = id$, then we define $M_{P'}$ as the permutation from K_9 or K_{10}, see Example 23 (3). The permutations from K_9 and K_{10} are 4-cycles, so the composition

$$g^{-1}M_{P'}gM_P = g^{-1}M_{P'}g$$

also is a 4-cycle.

(K_2). If $M_P = (13)(24)$, then we take $M_{P'} = (1'2'3'4') \in K_1$, see Example 23 (1). The composition

$$g^{-1}M_{P'}gM_P = (1234)(13)(24) = (1432)$$

is a 4-cycle.

(K_4). If $M_P = (14)(23)$, then we take $M_{P'} = (2'4') \in K_6$, see Example 23 (3). The composition

$$g^{-1}M_{P'}gM_P = (24)(14)(23) = (1234)$$

is a 4-cycle.

(K_5). If $M_P = (12)(34)$, then we take $M_{P'} = (2'4') \in K_6$, see Example 23 (3). The composition

$$g^{-1}M_{P'}gM_P = (24)(12)(34) = (1432)$$

is a 4-cycle.

(K_6). If M_P belongs to K_6, then we define $M_{P'}$ as the permutation from K_4 or K_5 (see Example 23 (2)) and come to the cases (K_4) and (K_5).

(K_9). If M_P belongs to K_9, then we define $M_{P'}$ as the permutation from K_0 (see Example 23 (1)) and come to the case (K_0).

(K_{10}). If M_P belongs to K_{10}, then we define $M_{P'}$ as the permutation from K_0 (see Example 23 (1)) and come to the case (K_0).

(K_{11}). If $M_P = (234)$, then we take $M_{P'} = (1'2'3'4') \in K_1$, see Example 23 (1). The composition

$$g^{-1}M_{P'}gM_P = (1234)(234) = (1243)$$

is a 4-cycle.

(K_{12}). If $M_P = (143)$, then we take $M_{P'} = (2'4') \in K_6$, see Example 23 (3). The composition

$$g^{-1}M_{P'}gM_P = (24)(143) = (1243)$$

is a 4-cycle.

Therefore, any z-homogeneous triangulation Γ of the surface S can be modified to other z-homogeneous triangulation of S containing less number of zigzags than Γ. Step by step, we come to a z-homogeneous and z-knotted triangulation of S. Since each single gluing reduce the number of zigzags in a z-orientation by at least one, we need at most $|\tau| - 1$ steps.

6.5 Appendix

Table 1 contains all calculations for elements of S_4 used in Subsection 6.2.

M_P	M_P^{-1}	sM_P	tM_P^{-1}	stM_P^{-1}	sM_Ps	$tM_P^{-1}t$	$stM_P^{-1}st$
id	id	(13)(24)	(12)(34)	(14)(23)	id	id	id
(34)	(34)	(1324)	(12)	(1423)	(12)	(34)	(12)
(23)	(23)	(1342)	(1243)	(14)	(14)	(14)	(23)
(234)	(243)	(132)	(123)	(142)	(124)	(134)	(123)
(243)	(234)	(134)	(124)	(143)	(142)	(143)	(132)
(24)	(24)	(13)	(1234)	(1432)	(24)	(13)	(13)
(12)	(12)	(1423)	(34)	(1324)	(34)	(12)	(34)
(12)(34)	(12)(34)	(14)(23)	id	(13)(24)	(12)(34)	(12)(34)	(12)(34)
(123)	(132)	(142)	(143)	(124)	(134)	(124)	(234)
(1234)	(1432)	(1432)	(13)	(24)	(1234)	(1234)	(1234)
(1243)	(1342)	(14)	(14)	(1243)	(1342)	(1243)	(1342)
(124)	(142)	(143)	(134)	(243)	(234)	(123)	(134)
(132)	(123)	(234)	(243)	(134)	(143)	(142)	(243)
(1342)	(1243)	(23)	(23)	(1342)	(1243)	(1342)	(1243)
(13)	(13)	(24)	(1432)	(1234)	(13)	(24)	(24)
(134)	(143)	(243)	(132)	(234)	(123)	(234)	(124)
(13)(24)	(13)(24)	id	(14)(23)	(12)(34)	(13)(24)	(13)(24)	(13)(24)
(1324)	(1423)	(34)	(1324)	(34)	(1423)	(1423)	(1324)
(1432)	(1234)	(1234)	(24)	(13)	(1432)	(1432)	(1432)
(142)	(124)	(123)	(234)	(132)	(243)	(132)	(143)
(143)	(134)	(124)	(142)	(123)	(132)	(243)	(142)
(14)	(14)	(1243)	(1342)	(23)	(23)	(23)	(14)
(1423)	(1324)	(12)	(1423)	(12)	(1324)	(1324)	(1423)
(14)(23)	(14)(23)	(12)(34)	(13)(24)	id	(14)(23)	(14)(23)	(14)(23)

Table 1

7 Triangulations with a z-monodromy of the same type for all faces

In this section, we return to z-monodromies of faces. We want to examine the existence of triangulations with the same type of z-monodromy for all faces.

For the z-monodromies (M3), (M4), (M5) and (M7) the situation is clear: the required triangulations are bipyramids, see Subsection 3.4. More precisely, the z-monodromy of every face in the bipyramid BP_n is of type

- (M3) if $n = 2k + 1$ and k is odd,
- (M4) if $n = 2k + 1$ and k is even,
- (M7) if $n = 2k$ and k is odd,
- (M5) if $n = 2k$ and k is even.

An example of a triangulation where all faces have the z-monodromy of type (M6) was constructed in [31] in connection with studying z-oriented triangulations with all faces of type I (Subsection 7.1).

For (M1) and (M2) this problem was investigated in [24] and the answer is negative: for these z-monodromies such triangulations does not exist. This follows from the more general result which states that for $i = 1, 2$ faces with the z-monodromy of type (Mi) form a forest (Subsections 7.2 – 7.4). However, there is a z-knotted triangulation (the connected sum of bipyramids) where the z-monodromy of each face is of type (M1) or (M2).

7.1 A triangulation where all faces have the z-monodromy of type (M6)

Our construction will be based on the following technical facts.

Proposition 6. *Let Γ be a triangulation and let F be a face of Γ. If M_F is* (M6)*, then F is of type I for any z-orientation of Γ.*

Proof. Let e_1, e_2, e_3 be consecutive oriented edges of the face F. If the z-monodromy of F is (M6), i.e.

$$M_F = (-e_1, e_3, e_2)(-e_2, -e_3, e_1)$$

then there are precisely two zigzags (up to reversing) containing F

$$\dots, e_1, e_2, \dots, -e_1, -e_3, \dots \quad \text{and} \quad \dots, e_2, e_3, \dots$$

The non-oriented edge corresponding to the oriented edges e_1 and $-e_1$ is passed in two different directions by the same zigzag. Thus, this edge is of type I for any orientation of the zigzag and F is of type I for any z-orientation. $\square$

Lemma 12. *Suppose that τ is a z-orientation of a triangulation Γ. Let F be a face in (Γ, τ) such that there are precisely two zigzags from τ which contain edges from F. Then the following assertions are fulfilled:*

(1) *There is a unique edge e belonging to F which occurs in one of these zigzags twice,*

(2) *The type of e does not depend on the choice of z-orientation,*

(3) *If e is of type I, then M_F is* (M6)*. If e is of type II, then M_F is* (M7)*.*

Proof. (1). Each face occurs precisely thrice (as a pair of its adjacent edges) in zigzags from τ. Since there are precisely two zigzags from τ passing through F, one of these zigzags passes through it once and the second twice.

(2). A zigzag can pass through the edge e twice either in two different directions (type I) or in the same direction (type II). The reversing of zigzag does not change the type of e, so the type of this edge is the same for any z-orientation of Γ.

(3). The z-monodromy of F is (M6) or (M7) by Remark 2. If the z-monodromy is (M6), then Proposition 6 gives the claim. Consider the case when M_F is of type (M7). Let e_1, e_2, e_3 be consecutive edges of F such that

$$M_F = (e_1, e_2)(-e_1, -e_2).$$

Thus, the face F occurs twice in the zigzag

$$\dots, e_2, e_3, \dots, e_3, e_1, \dots$$

and the edge e_3 is of type II for any z-orientation. $\square$

Using Lemma 12 we construct a class of toric triangulations, where the z-monodromy is of type (M6) for all faces.

Example 24. Let n, m be odd numbers not less than 3. Denote by Γ_0 a $n \times m$ grid with the opposite sides identified. Then Γ_0 is naturally embedded in a torus. Let $\Gamma = \mathrm{T}(\Gamma_0)$, see Fig. 21 for the case $n = m = 3$.

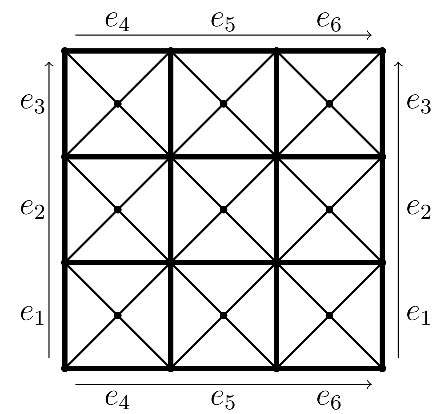

Figure 21

Every zigzag of Γ gives a band formed by n or m squares from the grid, see Fig. 22 for a band consisting of 5 squares.

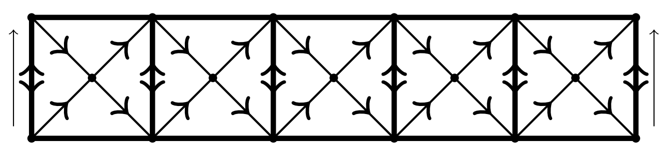

Figure 22

Note that the edges common for two consecutive squares from the grid are passed twice and they are of type I for any z-orientation (these edges are marked by the bold line in Fig. 22). Thus, each edge of the subgraph Γ_0 is of type I and all faces of Γ are of type I (for any z-orientation). The remaining edges from the interior of the band are passed by the zigzag once, so each edge incident to a vertex in the interior of a square occurs once in two different zigzags. Therefore, any face of Γ is passed by precisely two zigzags up to reversing. By Lemma 12, z-monodromies of all faces of Γ are of type (M6).

7.2 The z-monodromies (M1) and (M2)

Let Γ be a triangulation of a surface S. Recall that the dual graph Γ^* is a graph whose vertices are faces of Γ and whose edges are formed by pairs of distinct faces intersecting in an edge.

Let G_i be the subgraph of Γ^* such that the set of vertices of G_i consists of all faces of Γ whose z-monodromies are of type (Mi) and two vertices of G_i are adjacent if they are adjacent in Γ^*. Theorem 9 describe the subgraphs G_1 and G_2, i.e. the subgraphs determined by all faces with z-monodromy of type (M1) and (M2), respectively. This result does not require z-knottedness and it concerns to an arbitrary triangulation.

Theorem 9 ([24]). *The graphs* G_1 *and* G_2 *are forests.*

A direct consequence of the theorem is that triangulations with z-monodromy of type (Mi) for all faces do not exist for $i = 1, 2$.

Let Γ and Γ' be z-knotted triangulations containing faces F and F' (respectively) such that their z-monodromies are not identity, i.e. the z-monodromies of F and F' are not of type (M1). By Theorem 5, there is a special homeomorphism $g : \partial F \to \partial F'$ such that the connected sum $\Gamma \#_g \Gamma'$ is z-knotted. It follows from Theorem 9 that each triangulation contains a face whose z-monodromy is not of type (M1). Thus, we get the following corollary.

Corollary 3. *For any z-knotted triangulations Γ and Γ' there are faces F and F' in Γ and Γ' (respectively) and a special homeomorphism $g : \partial F \to \partial F'$ such that the connected sum $\Gamma \#_g \Gamma'$ is a z-knotted triangulation.*

7.3 Proof: the graph G_1 is a forest

The *face shadow* of a zigzag $Z = \{e_1, \dots, e_n\}$ is a cyclic sequence of faces $F_1, \dots, F_n$, where F_i is the face containing the edges e_i and e_{i+1}.

Lemma 13. *Let F be a face belonging to the face shadow $F_1, \dots, F_n$ of a certain zigzag. Then there are at most three distinct indices i such that $F_i = F$. If our triangulation is locally z-knotted for F, then there are precisely three such i.*

Proof. For every edge $e \in \Omega(F)$ we denote by $Z(e)$ the zigzag containing the sequence $e, D_F(e)$. Observe that $Z(e') = Z(e)^{-1}$ if $e' = -D_F(e)$. Also, it can be happened that $Z(e) = Z(e')$ for some distinct $e, e' \in \Omega(F)$. This means that $\mathcal{Z}(F)$ contains at most three pairs of zigzags Z, Z^{-1}. Therefore, if $F_1, \dots, F_n$ is the shadow of a zigzag from $\mathcal{Z}(F)$ and $F_i = F$ for four distinct indices i, then this zigzag is self-reversed which is impossible. In the case when the triangulation is locally z-knotted in F, for any two $e, e' \in \Omega(F)$ we have $Z(e) = Z(e')$ or $Z(e) = Z(e')^{-1}$ which implies the second statement. $\square$

Lemma 14. *Let F and F' be adjacent faces whose z-monodromies both are of type* (M1)*. Then there is a unique* (*up to reversing*) *zigzag whose face shadow contains F and F'. This face shadow is a cyclic sequence of type*

$$\dots, F, F', \dots, F', \dots, F', F, \dots, F, \dots$$

($\textit{see}$ Fig. 23). *The reversed sequence*

$$\dots, F', F, \dots, F, \dots, F, F', \dots, F', \dots$$

is the face shadow of the reversed zigzag.

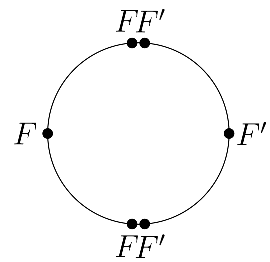

Figure 23

Proof. Let x, y, z and t, y, z be the vertices of F and F' (respectively) and let

$$e_1 = yz, \quad e_2 = zx, \quad e_3 = xy, \quad e_2' = zt, \quad e_3' = ty,$$

see Fig. 24. The intersection of $\Omega(F)$ and $\Omega(F')$ is $\{e_1, -e_1\}$.

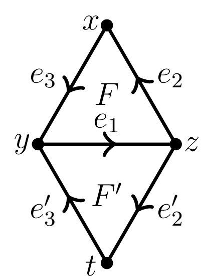

Figure 24

The z-monodromies M_F and $M_{F'}$ both are of type (M1), so Γ is locally z-knotted for F and F'. Since the faces F, F' are adjacent, we have

$$\mathcal{Z}(F) = \mathcal{Z}(F') = \{Z, Z^{-1}\}.$$

Suppose that Z is the zigzag containing the sequence e_3, e_1, e_2'.

Let e be the first edge from $\Omega(F) \cup \Omega(F')$ occurring in Z after this sequence. If e is an element from $\Omega(F)$, then it coincides with $M_F(e_1) = e_1$. This is impossible since we can come to e_1 by a zigzag only through an element of $\Omega(F)$ or $\Omega(F')$ different from e_1. Thus, e is an element from $\Omega(F')$ and $e = M_{F'}(e'_2) = e'_2$. The next edge of Z is $D_{F'}(e'_2) = e'_3$ and Z is a cyclic sequence

$$\ldots, e_3, e_1, e'_2, X, e'_2, e'_3, \ldots,$$

where X is a sequence of edges which does not contain elements from $\Omega(F) \cup \Omega(F')$. Similarly, we establish that e'_3 is the first edge from $\Omega(F) \cup \Omega(F')$ occurring in Z after the sequence e'_2, e'_3. The next two edges of Z are $D_{F'}(e'_3) = e_1$ and $D_F(e_1) = e_2$. Therefore, Z is a cyclic sequence

$$\ldots, \underbrace{e_3, e_1, e'_2}_{F,F'}, X, \underbrace{e'_2, e'_3}_{F'}, Y, \underbrace{e'_3, e_1, e_2}_{F',F}, \ldots,$$

where Y is a sequence of edges which does not contains elements of $\Omega(F) \cup \Omega(F')$. Since the next two edges from $\Omega(F) \cup \Omega(F')$ contained in the zigzag Z are $M_F(e_2) = e_2$ and $D_F(e_2) = e_3$, the second part of Lemma 13 gives the claim. $\square$

Now, we can show that G_1 is a forest. Suppose that $n \geq 4$ and $F_1, F_2, \ldots, F_n = F_1$ is a simple cycle in G_1. Since for each $i = 1, \ldots, n-1$ the triangulation is locally z-knotted for F_i and faces F_i, F_{i+1} are adjacent, we have $\mathcal{Z}(F_i) = \mathcal{Z}(F_{i+1})$. Thus,

$$\mathcal{Z}(F_1) = \cdots = \mathcal{Z}(F_{n-1}) = \{Z, Z^{-1}\}.$$

By Lemma 14, the faces F_1 and F_2 occur in the face shadow of Z or Z^{-1} as follows

$$\ldots, F_2, F_1, \ldots, F_1, \ldots, F_1, F_2, \ldots, F_2, \ldots;$$

without loss of generality we assume that this is the face shadow of Z. The faces F_2 and F_3 are adjacent and there are four possibilities for F_3 to occur in Z, see Fig. 25.

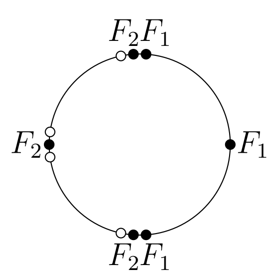

Figure 25

By Lemma 14, one of the following possibilities is realized for F_3:

$$\dots, F_3, F_2, F_1, \dots, F_1, \dots, F_1, F_2, \dots, F_2, F_3, \dots, F_3, \dots$$

or

$$\dots, F_2, F_1, \dots, F_1, \dots, F_1, F_2, F_3, \dots, F_3, \dots, F_3, F_2, \dots$$

(see Fig. 26 (a) and Fig. 26 (b), respectively).

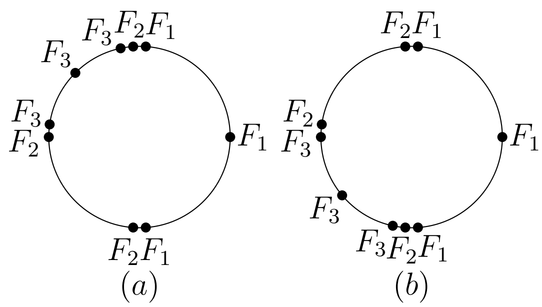

Figure 26

The faces F_3 and F_4 are adjacent and (by Lemma 14) there are three occurrences of F_4 in the face shadow of Z after the three occurrences of F_1. Recursively, we establish that the same holds for each F_i where $3 \leq i \leq n$, i.e.

$$\dots, F_2, F_1, \dots, F_1, \dots, F_1, F_2, \dots, F_i, \dots, F_i, \dots, F_i, \dots.$$

If $F_i = F_n = F_1$, then the above contradicts the fact that the face shadow of Z contains precisely three occurrences of F_1. Therefore, G_1 does not contain cycles.

7.4 Proof: the graph G_2 is a forest

Lemma 15. *Let F and F' be adjacent faces whose z-monodromies are of type* (M2). *Then there is a unique (up to reversing) zigzag whose face shadow contains F and F'. This face shadow is a cyclic sequence of type*

$$\dots, F, F', \dots, F, \dots, F', F, \dots, F', \dots$$

(*see* Fig. 27). *The reversed sequence*

$$\dots, F', F, \dots, F,' \dots, F, F', \dots, F, \dots$$

is the face shadow of the reversed zigzag.

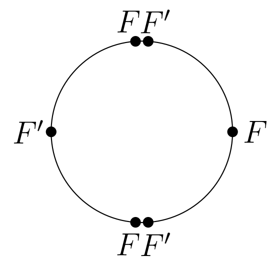

Figure 27

Proof. Suppose that F and F' are as in the proof of Lemma 14 (see Fig. 24). Since $M_F = D_F$ and $M_{F'} = D_{F'}$, the triangulation is locally z-knotted for F and F'. As in the proof of Lemma 14, we establish that

$$\mathcal{Z}(F) = \mathcal{Z}(F') = \{Z, Z^{-1}\}$$

and assume that Z contains the sequence e_3, e_1, e'_2.

Let e be the first edge from $\Omega(F) \cup \Omega(F')$ occurring in Z after this sequence. If e is an element from $\Omega(F')$, then

$$e = M_{F'}(e'_2) = D_{F'}(e'_2) = e'_3$$

and the next edge in the zigzag is $D_{F'}(e'_3) = e_1$. Hence $M_F(e_1) = e_1$. This is impossible, since

$$M_F(e_1) = D_F(e_1) = e_2.$$

Thus, e belongs to $\Omega(F)$ and we have $e = M_F(e_1) = e_2$. The next edge in the zigzag Z is $D_F(e_2) = e_3$ and Z is a cyclic sequence

$$\ldots, e_3, e_1, e'_2, X, e_2, e_3, \ldots,$$

where X is a sequence of edges which does not contain elements of $\Omega(F) \cup \Omega(F')$.

Let e' be the first edge from $\Omega(F) \cup \Omega(F')$ occurring in Z after the sequence e_2, e_3. If $e' \in \Omega(F)$, then

$$e' = M_F(e_3) = D_F(e_3) = e_1.$$

This is impossible, since we can come to e_1 by a zigzag only through an element of $\Omega(F)$ or $\Omega(F')$ different from e_1. Thus, $e' \in \Omega(F')$ and

$$e' = M_{F'}(e'_2) = D_{F'}(e'_2) = e'_3.$$

The next two edges in Z are $D_{F'}(e'_3) = e_1$ and $D_F(e_1) = e_2$. Therefore, Z is a cyclic sequence

$$\dots, \underbrace{e_3, e_1, e'_2}_{F,F'}, X, \underbrace{e_2, e_3}_{F}, Y, \underbrace{e'_3, e_1, e_2}_{F',F}, \dots,$$

where Y is a sequence of edges which does not contains elements of $\Omega(F) \cup \Omega(F')$. Since the next two edges from $\Omega(F)\cup\Omega(F')$ contained in the zigzag Z are $M_{F'}(e_1) = e'_2$ and $D_{F'}(e'_2) = e'_3$, the second part of Lemma 13 gives the claim. □

Now, we establish that G_2 is a forest (in this case we use more complicated arguments then for G_1). Suppose that $n \geq 4$ and $F_1, F_2, \dots, F_n = F_1$ is a simple cycle in G_2. As in the previous subsection, the triangulation is locally z-knotted for each F_i and

$$\mathcal{Z}(F_1) = \dots = \mathcal{Z}(F_{n-1}) = \{Z, Z^{-1}\}.$$

By Lemma 15, the faces F_1 and F_2 occur in the face shadow of Z or Z^{-1} as follows

$$\dots, F_1, F_2, \dots, F_1, \dots, F_2, F_1, \dots, F_2, \dots;$$

without loss of generality we suppose that this is the face shadow of Z.

(1). Let $\dots, F_1, F_2, \dots, F_n$ be consecutive faces in the face shadow of Z. The faces F_2, F_3 are adjacent and, by Lemma 15, the face shadow of Z is

$$\dots, F_1, F_2, F_3, \dots, F_1, \dots, F_2, F_1, \dots, F_3, F_2, \dots, F_3, \dots,$$

see Fig. 28 (a). For $n > 4$ we apply Lemma 15 to the adjacent faces F_{i-1}, F_i, where $4 \leq i \leq n-1$. Recursively, we establish that F_{n-1} occurs in the face shadow of Z as follows

$$\dots, F_1, \dots, F_{n-1}, F_1, \dots, F_1, \dots, F_{n-1}, \dots, F_{n-1}, \dots,$$

see Fig. 28 (b).

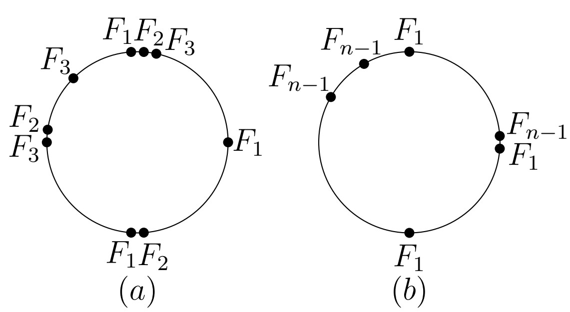

Figure 28

On the other hand, F_{n-1} is adjacent to $F_n = F_1$ and Lemma 15 guarantees that the face shadow of Z is as follows

$$\dots, F_1, \dots, F_{n-1}, F_1, \dots, F_{n-1}, \dots, F_1, F_{n-1}, \dots;$$

we get a contradiction.

(2). Suppose that $F_1, F_2, \dots, F_n$ are not consecutive faces in the face shadow of Z. Let k be the greatest number such that $F_1, \dots, F_k$ are consecutive faces in the face shadow of Z. Since the sequence F_1, F_2 is contained in the face shadow, we have $k \geq 2$. If $k = 2$, then (by Lemma 15) the face shadow of Z is a cyclic sequence

$$\dots, F_1, F_2, \dots, F_1, \dots, F_3, F_2, F_1, \dots, F_3, \dots, F_2, F_3, \dots,$$

see Fig. 29. Then F_1, F_2, F_3 are consecutive faces in the face shadow of the reversed zigzag Z^{-1}, which is impossible. Thus, $k \geq 3$.

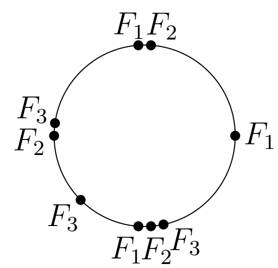

Figure 29

We apply Lemma 15 to the faces F_{k-1}, F_k and to the faces F_k, F_{k+1}. The face F_{k+1} does not occur in the face shadow of Z immediately after $F_1, \dots, F_k$, so the face shadow of Z is as follows

$$\dots, F_1, F_2, \dots, F_{k-1}, F_k, \dots, F_1, \dots, F_2, F_1, \dots, F_{k+1}, F_k, F_{k-1}, \dots, F_{k+1}, \dots, F_k, F_{k+1}, \dots,$$

see Fig. 30.

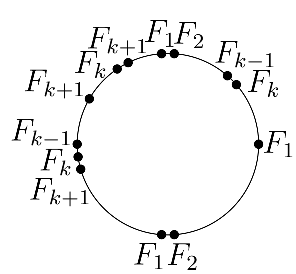

Figure 30

Finally, for each i such that $k < i \leq n$ the face F_i occurs in the face shadow of Z in the following way

$$\ldots, F_1, F_2, \ldots, F_1, \ldots, F_2, F_1, \ldots, F_i, \ldots, F_i, \ldots, F_i, \ldots;$$

if $F_i = F_n = F_1$, then it means that F_1 occurs in the face shadow of Z more than three times. This is impossible.

We come to a contradiction in both these cases. Therefore, G_2 does not contain cycles.

7.5 Two examples

In this subsection, we present two examples of graphs G_1 and G_2 in connected sums of bipyramids. The first example is simple. The second is interesting for the following reason: there is a z-knotted triangulation where the z-monodromy of each face is of type (M1) or (M2).

Before we give these examples, let us observe the following fact. Let Γ be locally z-knotted for a face F. Then the z-monodromy M_F is one of the types (M1)–(M4). If M_F is of type (M3) or (M4), then each zigzag from $\mathcal{Z}(F)$ passes through one of edges twice in the same direction and it passes twice through the remaining two edges in different directions, see Fig. 31 (a). Thus, the face F is of type I.

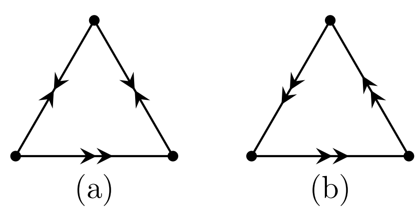

Figure 31

If M_F is of type (M1) or (M2), then each zigzag from $\mathcal{Z}(F)$ passes through each edge of F twice in the same direction, i.e. this zigzag passes twice through three elements from $\Omega(F)$ forming a cycle in D_F, see Fig. 31 (b). Thus, the face F is of type II.

As in the previous sections, let the consecutive vertices of the base of BP_n be denoted by $1, \ldots, n$ and the two remaining vertices by a and b. Similarly, let BP'_n be the n-gonal bipyramid whose consecutive vertices of the base are $1', \ldots, n'$ and let a', b' be the remaining two vertices.

Example 25. Consider the z-knotted bipyramids PB_3 and PB'_3. Their zigzags are the cyclic sequences of edges

$$\underbrace{12, 2b, b3, 31, 1a}_{A}, \underbrace{a2, 23, 3b, b1, 12}_{B}, \underbrace{2a, a3, 31, 1b, b2, 23, 3a, a1}_{C}$$

and

$$\underbrace{1'2', 2'b', b'3', 3'1', 1'a'}_{A'}, \underbrace{a'2', 2'3', 3'b', b'1', 1'2'}_{B'}, \underbrace{2'a', a'3', 3'1', 1'b', b'2', 2'3', 3'a', a'1'}_{C'},$$

respectively. Denote by D and D' the faces of BP_3 and BP'_3 whose vertices are $a, 1, 2$ and $a', 1', 2'$ (respectively). We glue these bipyramids using a special homeomorphism $g : \partial D \to \partial D'$ such that

$$g(a) = a', \quad g(1) = 1', \quad g(2) = 2'$$

and we obtain the connected sum $BP_3 \#_g BP'_3$ (see Fig. 32 and Example 12).

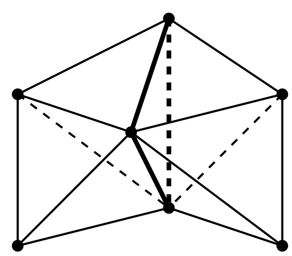

Figure 32

The connected sum is z-knotted and it has the unique zigzag (up to reversing)

$$Z = \{A, C'^{-1}, B, A', C^{-1}, B'\},$$

where C^{-1} and C'^{-1} are the sequences reversed to C and C' (respectively) and for any two consecutive parts X, Y in Z the last edge from X is identified with the first edge from Y. Let F and F' be the faces of $BP_3 \#_g BP'_3$ which contain $b, 1, 2$ and $b', 1, 2$, respectively. Example 12 shows that the z-monodromies of these faces are of type (M2). Each of the remaining eight faces contains one of the edges $23, 31, 23', 3'1$. The zigzag Z passes through each of these edges twice in different directions Thus, the faces containing one of these edges are of type I and their z-monodromies are of type (M3) or (M4). The subgraph G_2 in $BP_3 \#_g BP'_3$ is a linear graph P_2. The same is true for the connected sum of $(2k+1)$-gonal and $(2k'+1)$-gonal bipyramids, where k and k' are odd (see the details in Example 12).

Example 26. Consider 6-gonal bipyramids BP_6 and BP'_6, Each of them contains precisely two zigzags (up to reversing). The cyclic sequences

$$\underbrace{12, 2b, b3, 34, 4a, a5, 56, 6b, b1, 12}_{A}, \underbrace{2a, a3, 34, 4b, b5, 56, 6a, a1}_{B}$$

and

$$\underbrace{a2, 23, 3b, b4, 45, 5a, a6, 61, 1b, b2, 23, 3a, a4, 45, 5b, b6, 61, 1a}_{C}$$

are zigzags in BP_6. Similarly, the cyclic sequences

$$\underbrace{1'2', 2'b', b'3', 3'4', 4'a', a'5', 5'6', 6'b', b'1', 1'2'}_{A'}, \underbrace{2'a', a'3', 3'4', 4'b', b'5', 5'6', 6'a', a'1'}_{B'}$$

and

$$\underbrace{a'2', 2'3', 3'b', b'4', 4'5', 5'a', a'6', 6'1', 1'b', b'2', 2'3', 3'a', a'4', 4'5', 5'b', b'6', 6'1', 1'a'}_{C'}$$

are zigzags in BP'_6. Denote by D and D' the faces of BP_6 and BP'_6 which contain the vertices $a, 1, 2$ and $a', 1', 2'$, respectively. Let $g : \partial D \to \partial D'$ be the special homeomorphism such that

$$g(a) = 2',\ g(1) = a',\ g(2) = 1',$$

see Example 13. The connected sum $BP_6 \#_g BP'_6$ is z-knotted and the unique zigzag (up to reversing) is

$$Z = \{A, C'^{-1}, C^{-1}, A', B, B'\};$$

as in Example 25, for any two consecutive parts X, Y in Z the last edge from X is identified with the first edge from Y. We need the following observations about the edges of BP_6:

- Each of the edges $12, 34, 56$ occurs twice in the sequence A, B and each of the edges $23, 45, 61$ occurs twice in the sequence C.
- An edge e which contains a or b occurs in the sequence A, B if and only if $-e$ occurs in C.

The same statements are true for the edges of BP'_6. Thus, Z passes through every edge of $BP_6 \#_g BP'_6$ twice in the same direction. Then each face of the connected sum $BP_6\#_g BP_6'$ is of type II and its z-monodromy is of type (M1) or (M2). It is easy to check that the z-monodromies of the faces

$$a23, a34, a61, a13', b45, b56, 15'6', 126', b'3'4', b'4'5'$$

and the faces

$$a45, a56, 13'4', 14'5', b61, b12, b23, b34, ab'3', ab'2, b'26', b'5'6'$$

are of types (M1) and (M2), respectively. Therefore, the graph G_1 is a linear forest consisting of five P_2 and the graph G_2 is a linear forest consisting of two P_2 and two P_4.

Bibliography

[1] Bonnington C.P. , Conder M., Morton M. and McKenna P., *Embedding digraphs on orientable surfaces*, J. Combin. Theory Ser. B 85 (2002), 1–20.

[2] Bonnington C.P., Hartsfield N. and Širáň J. *Obstructions to directed embeddings of Eulerian digraphs in the plane*, European J. Combin. 25 (2004), 877–891.

[3] Brinkmann G., Dress, A. W. M., *PentHex puzzles. A reliable and efficient top-down approach to fullerene-structure enumeration*, Adv. Appl. Math. 21 (1998), 473–480.

[4] Chen Y., Gross J. L., and Hu X., *Enumeration of digraph embeddings*, European J. Combin. 36 (2014), 660–678.

[5] Coxeter H.S.M., *Regular polytopes*, Methuen, 1948.

[6] Crapo H., Rosenstiehl P., *On lacets and their manifolds*, Discrete Math. 233 (2001), 299–320.

[7] Deza M., Dutour Sikirić M., Shtogrin M., *Geometric Structure of Chemistry-relevant Graphs: zigzags and central circuit*, Springer 2015.

[8] Deza M., Pankov M., *Zigzag Structure of Thin Chamber Complexes*, Discrete Comput. Geom. 59 (2018), 363–382.

[9] Farr G. E., *Minors for alternating dimaps*, Q. J. Math. 69 (2018), 285–320.

[10] de Fraysseix H., Ossona de Mendez P., *On a Characterization of Gauss Codes*, Discrete Comput. Geom. 22 (1999), 287–295.

[11] Godsil C., Royle G., *Algebraic Graph Theory*, Graduate Texts in Mathematics 207, Springer 2001.

[12] Graver J. E., Watkins M. E., *Locally finite, planar, edge-transitive graphs*, Mem. Am. Math. Soc. 601 (1997), 75 p.

[13] Hjelle Ø., Dæhlen M., *Triangulations and Applications*, Springer 2006.

[14] Jones G. A., Thornton J. S., *Operations on maps, and outer automorphisms*, J. Comb. Theory, Ser. B 35 (1983), 93–103.

[15] Kinsey L. C., *Topology of Surfaces*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1993.

[16] Lins S., *Graphs of Maps*, arXiv:math/0305058, PhD Thesis.

[17] Lins S., *Graph-encoded maps*, J. Combin. Theory, Ser. B 32 (1982), 171–181.

[18] Lins S., Oliveira-Lima E., Silva V., *A homological solution for the Gauss code problem in arbitrary surfaces*, J. Combin. Theory, Ser. B 98 (2008), 506–515.

[19] Lins S., Richter B., Shank H., *The Gauss code problem off the plane*, Aequationes Math. 33 (1987) 81–95.

[20] Lovasz L., Marx M. L., *A forbidden subgraph characterization of Gauss codes*, Bull. Amer. Math. Soc. 82 (1976) 121–122.

[21] McCourt T. A., *Growth rates of groups associated with face 2-coloured triangulations and directed Eulerian digraphs on the sphere*, Electron. J. Comb. 23 (2016), P1.51, 23 p.

[22] Mohar B., Thomassen C., *Graphs on Surfaces*, The Johns Hopkins University Press 2001.

[23] Pankov M., Tyc A., *Connected sums of z-knotted triangulations*, Euro. J. Comb. 80 (2019), 326–338.

[24] Pankov M., Tyc A., *On two types of z-monodromy in triangulations of surfaces*, Discrete Math. 342 (2019), 2549–2558.

[25] Pankov M., Tyc A., *z-Knotted Triangulations of Surfaces*, Discrete Comput. Geom. 66 (2021), 636–658.

[26] Read R. C., Rosenstiehl P., *On the Gauss crossing problem*, Colloq. Math. Soc. Janos Bolyai, v. 18, Combinatorics, Keszthely, Hungary, 1976, pp. 843–876.

[27] Rosenstiehl P., *Solution algébrique du problème de Gauss sur la permutation des points d'intersection d'une ou plusieurs courbes fermées du plan*, C.R. Acad. Sci. Paris, Sér A 283 (1976) 551–553.

[28] Rosenstiehl P., *A new proof of the Gauss interlace conjecture*, Adv. Appl. Math. 23 (1999) 3–13.

[29] Shank H., *The theory of left-right paths* in Combinatorial Mathematics III, Lecture Notes in Mathematics 452, Springer 1975, 42–54.

[30] Tyc A., *Z-knotted and z-homogeneous triangulations of surfaces*, Discrete Math. 344 (2021), 112405.

[31] Tyc A., *Z-oriented triangulations of surfaces*, Ars Math. Contemp., published online, https://doi.org/10.26493/1855-3974.2242.842

[32] Williams G., *Petrie schemes*, Can. J. Math. 57 (2005), no. 4, 844–870.

[33] Wilson R. J., *Introduction to Graph Theory*, Longman 1996.

[34] Wilson S. E., *Operators over regular maps*, Pac. J. Math. 81 (1979), 559–568.