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Evaluation of the doctoral thesis *Ranks of tensors, related varieties and rank additivity property for small classes* by Filip Rupniewski

The presented thesis is on the border of at least two areas of mathematics: multilinear algebra with the emphasis on the tensor calculus and algebraic geometry which manifests itself through considered secant and cactus varieties and is also present in applied methods and tools. The thesis is rendered by a Macaulay2 code which underlays computational aspects of discussed subjects.

The theory studied in the thesis has its roots in times of Diophantus (around the year 250). A more modern motivation for various questions discussed here is provided by the celebrated Lagrange Four Square Theorem (around the year 1770) and a question of Waring about natural generalizations asked in the same period. Over the years many excellent mathematicians contributed to the subject in its original setting, i.e., in the number theory, suffices it to mention Euler, Hilbert, Dickson, Mahler and many others. The study in the course has been translated in algebraic geometry for properties of secant varieties of Veronese varieties and generalized subsequently to tensors and projective varieties. It is worth to note that the Waring problem for homogeneous polynomials has been solved only in 1995 by Alexander and Hirschowitz.

The presented thesis is a small but fine contribution to the Waring's problem in its generalized setting.

Turning to more details the main objects of interest in the thesis are:

- tensor rank;
- tensor border rank;
- cactus rank;
- cactus border rank;

and variations thereof. The thesis consists of four chapters. Chapter 1 is of the introductory nature. It contains main motivations and explains the notions mentioned above. It reveals also, in Section 1.2.5, the main problem for the whole theory, namely the failure of the Strassen Additivity Conjecture as discovered by Shitov in 2019 (the result was circulated already around 2017).

Chapter 2 explains in more detail main techniques and tools applied in the thesis. These include slice technique and conciseness, projections and various Apolarity Lemmas. Even though Section 2.2.2 has a slightly speculative character, it deserves attention since it explains in detail the concept of what the author calls "hook"-shaped spaces. This chapter concludes with detailed study of two families of subvarieties with prescribed Hilbert functions.

Chapter 3 revolves around additivity proofs of tensor rank and tensor border rank in some new cases. The key ideas here are the minimal decomposition of tensors and detailed classification of seven types of matrices appearing in this situation. The analysis culminates in the proof of rather technical Theorem 1.2.5.7 which is one of the most important contributions of the thesis. In the case of tensor border rank the author speculates a bit about possible counterexamples to the additivity in case of 3-way tensors on a 5-dimensional complex vector space. This part of the thesis builds on a joint paper with the advisor and Elisa Postinghel.

The final chapter 4 contains results obtained jointly with Tomasz Mańdziuk and Maciej Gałązka. These results concern a very specific class of examples, namely the 14-th cactus varieties of some Veronese varieties. These results are so technical that it's difficult to sum them up here in one or two sentences.

Attached Macaulay2 code renders in a nice way presented results. This is certainly an added value of the thesis, especially taking into account that the code can be easily adjusted either to other symbolic computations programs, e.g, Singular and, more importantly, to other situations occurring in researcher's lab.

Even though many of results contained in the thesis have been obtained in joint works with various authors, there is no doubt about substantial contribution of Mr. Rupniewski to all of them. Showing the ability of cooperation in national and international teams is, in my opinion, an additional plus to be noted on Mr. Rupniewski account, as current research also in mathematics builds upon team work and exchange of ideas rather than on works of a lonely scientist spending hours in an obscure corner of a library, which was considered a standard 30 years ago, and to some expend is still prized in certain circles.

The thesis presented by Mr. Rupniewski is a piece of good mathematical work. It contains new results, some new techniques, explores available techniques beyond limits known before and proves author's ability to work in a competent and creative way on a mathematical problem. Its level places it somewhere in the middle among PhD thesis from universities in US, Spain and German I have refereed recently. Thus **I recommend to accept the thesis** and to take further steps in the whole procedure.

In Polish: Stwierdzam, że przedłożona rozprawa spełnia formalne i zwyczajowe wymagania stawiane pracom doktorskim z matematyki i rekomenduję dopuszczenie mgr Rupniewskiego do dalszych kroków przewodu doktorskiego.

Tomasz Szemberg

Signed by /
Podpisano przez:

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