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REPORT on the THESIS of F. RUPNIEWSKI

The thesis concerns the geometric and algebraic study of spaces of tensors. The topic is very hot, due to the wide series of applications that tensors have in many modern sciences and technology, ranging from Signal Theory to Quantum Physics and Biology. Spaces of tensors are actually one of the most relevant subjects in which even deep advances in theoretical aspects have good chances to be immediately applicable to concrete problems. Moreover, the advances often require the most sophisticated tools from many fields of mathematics, including projective algebraic geometry.

As in the case of matrices, even for general tensors there is a notion of rank which incorporates their fundamental aspects and properties. While for matrices we have efficient methods that compute the rank up to a satisfactory approximation, nothing similar is known when the dimension of the tensor is bigger than 2. New results in the topic, as well as new procedures for the computation, promise to produce important advances in our knowledge.

The candidate focus mainly on two problems, related to the computation of the rank of a given tensor, or of a given space of tensors, and to the computation of its variational equivalent: the border rank.

The first problem concerns the additivity of the rank for sum of tensors defined over disjoint subspaces (or disjoint sets of variables).

A very recent example, found by Y. Shitov, shows that the additivity property can fail. Yet, the general feeling is that the failure should be bounded to specific, quite rare, types of tensors. Then researchers are looking for conditions that make the additivity hold. The list of cases in which such conditions are known is still quite poor.

In chapter 2 of the Thesis, the candidate produces a new list of cases for which the additivity problem has a general positive answer. The new cases concern tensors living in the product of three vector spaces, and consider mainly situations in which the rank of one of the summands is not much bigger than its maximal dimension. The additivity of the rank follows from a clever investigation of the interaction of the slicing process, which associates to a three-dimensional tensor a space of matrices, and the addition of ranks. The results of the candidate are new, to my knowledge, and the techniques introduced for the proofs can be probably extended, in future investigations, to a larger set of tensors.

The second problem concerns the distinction between the border rank of a tensor and its cactus rank, i.e. the rank defined by a non-necessarily reduced decomposition.

In many computational and algebraic procedures it is quite difficult to separate the variety of tensors of given border rank r from the variety of tensors having cactus rank r, which is realized only by non-smoothable sets of points. Indeed, methods that produce equations for the loci of tensors of given border rank (secant varieties) are often unable to distinguish between the two components.

The candidate considers spaces of symmetric tensors, for which one can properly use the associated apolar ideal to investigate the structure of the relative sub-loci. Taking advantage from the fact that a description of some components of the non-smoothable locus is now available in the literature, the candidate produces a concrete algorithm that, in two specific cases, guarantees that a given symmetric tensor belongs to the border rank component, and not to the smoothable component, in the locus defined by some known equations for the border rank. The achievement represents a completely new type of results. Extensions of the method for other types of tensors rely on finding a precise description of the corresponding varieties of schemes of finite length in projective spaces.

The Thesis, which certainly contains several technicalities, is sufficiently clear and well written. As far as I could check, the arguments are complete.

From a scientific point of view, the achievements of the Thesis are absolutely relevant, and worthy of publications in journals of very good level. Indeed, most of the results have been already published (SIMAX 2020) or presented in preprints (arXiv) in joint papers of the candidate.

I appreciated that, when possible, the candidate makes a precise analysis of the ground fields in which the new results hold, including non-algebraically closed fields and fields of positive characteristic.

I include a short list of minor remarks and suggestions.

In conclusion, I express a very favorable advice in support of awarding the candidate F. Rupniewski with the degree of Doctor of Mathematical Sciences.

I certainly found the thesis to be in the top 20% of Ph.D.'s that I ever reviewed, so that I also strongly suggest to classify it as "outstanding".

Siena 30/12/21

Luca Chiantini

(Luca Chiantini)

REMARKS

- The technique of slicing a tensor is classical. The relations between the rank of a tensor and the rank of the space defined by its slices were found by Terracini (Sulla rappresentazione delle coppie di forme ternarie mediante somme di potenze di forme lineari. Ann. Mat. Pura Appl. XXIV (1915), 91-100). The candidate seems to ignore Terracini's paper and the ample subsequent literature based on the "second Terracini's Lemma". See e.g. the first part of Lemma 2.1.4.1.
- At page 17 the candidate introduces the notation, but some piece is missing, as the reference to the symbols introduced in section1.3.1 (T_1) or the powers ($L_1^{|d|}$…).
- I do not see why the last statement of Lemma 2.2.1.5 follows from Corollary 2.2.1.4. If E'={0} then e'=0 can be smaller than R(W')-dimW', which can be 1.
- The notation in Definition 2.2.2.1 is not coordinate-free.
- In Definition 2.4.0.1, what is $\mathbf{k}_{dp}$?
- The title of Section 2.4.1 should be "X-rank and X-border rank". A similar remark holds for Section 2.4.2.
- At the beginning of page 37 "The" is probably missing.
- In Lemma 2.5.0.3, does the ideal I need to be saturated?
- In the proof of Lemma 2.5.0.9 part (iii) it is not repeated that Θ_I has degree i. Also, it is not clear to which subspaces ρ,ρ' belong.
- Definition 2.5.0.14 should be anticipated, for the Hilbert scheme has been already mentioned at page 33.
- In the formula in the proof of Lemma 2.6.0.1 the second inequality is not clear.
- I do not see the role of V in the proof of lemma 2.6.0.2.
- In the proof of Lemma 2.6.0.5 it is important to observe that the upper horizontal map of the diagram is surjective. The same remark holds for the proof of lemma 2.7.0.7.
- In the proof of Lemma 2.6.0.5 the equality $\mathcal{U}^0 = \mathcal{U} \cap \mathcal{B}$ is not clear.
- At page 60, line 6, I imagine that "resp." means "with respect to Prime or Bis". It is not clear, however, why the repletion does not depend on the ordering of the variables.
- In the proof of Lemma 3.1.3.4, apparently at a certain point the role of S,S' is taken by W,W'.
- At line -2 of page 66 should be "… it follows…".
- Page 69 line 15. The first sentence of the paragraph is not concluded.
- Page 70 line 15 should start with "It results …". In the middle it should be "…it follows…".
- Page 84 line 5. It is not clear why T*/J has an (r-1,n+1)-standard Hilbert function.
- Starting from page 89, some notation is hard to follow since it refers to symbols (like e.g. *C*) which were introduced many pages above.
- Page 91 line -17. It is unclear why κ_{13} is equal to σ_{13}.