Mathematics
UNIVERSITY OF TORONTO

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Report on the Thesis
'Some Applications of Set Theory in Banach Spaces and Operator Algebras'
by
Damian Glodkowski

The thesis presents several important advances in Banach spaces and Operator algebras using methods of set theory and set-theoretic topology. They are naturally split into four separate chapters each devoted to a group of closely related problems. Chapter 2 studies the standard cardinal invariants of the σ-ideal of subsets of a given Banach space X of dimension > 1 generated by the family of hyperplanes of X. The most interesting of these is the number $\mathfrak{cov}(X)$, the minimal number of hyperplanes needed to cover the Banach space X. For example, one of the main results of this chapter is that this number has the minimal value ω_1 assuming various additional set-theoretic principle such as CH, PFA, or the statement that compact spaces with small diagonal are metrizable. The second interesting invariant of the σ-ideal generated by the hyperplanes of X is the number $\mathfrak{non}(X)$, the minimal cardinality of a subset of X that cannot be covered by countably many of hyperplanes of X. Another important result of Chapter 2 is that the set-theoretic assumption GCH or PFA[1] we have that $\mathfrak{non}(X)$ is either equal to the density $\mathrm{dens}(X)$ of X or its successor $\mathrm{dens}(X)^+$ depending on whether this number has uncountable of countable cofinality, respectively. Given the well known fact that the isomorphism between the Banach spaces $C(K)$ and $C(L)$ does not guarantee that the compacta K and L are of the same dimension (a consequence, for example, of a classical theorem of Miljutin), the Chapter 3 considers the problem of the existence of a compact space K and $n \in \omega \cup \{\infty\}$ such that for any other compact space L, if the Banach space $C(L)$ is isomorphic to $C(K)$ then L must have dimension n. The main result of this chapter is a remarkable construction of such compact space K (one for every $n \in \omega \cup \{\infty\}$) using the

[1] The result is actually stated with MM in place of PFA but it is clear from the proof that only some consequences of PFA are used.

Bahen Center, 40 St. George St., Room 6290, Toronto, ON M5S2E4 Canada
Tel: +1 416 978-3323 • Fax: +1 416 978-4107 • www.math.utoronto.ca

set-theoretic principle $\diamond$. In my opinion, just results of Chapters 2 and 3 are sufficient for an excellent PhD thesis, but the Thesis has two more Chapters treating some other classical problems in Banach spaces and Operator algebras. For example, the Chapter 4 gives a subtle forcing construction a Boolean space K for which the corresponding Banach space $C(K)$ has the Grotendieck property but not the Nykodim property. Distinguishing these two classical properties of Banach spaces has been a theme of an active research over the last forty years. The forcing construction can be seen as substantial modification of an old CH-construction of Talagrand and the result bares on the natural question about how essential is the Continuum Hypothesis CH for the distinction between the Grothendieck and the Nykodim properties. The last Chapter 5 belongs to by now a large body of work that studies the analogies between the quotient algebra $\mathcal{P}(\omega)/Fin$ and the Calkin algebra $\mathcal{Q}(\ell_2) = \mathcal{B}(\ell_2)/\mathcal{K}(\ell_2)$. Since the CH guarantees the universality of $\mathcal{P}(\omega)/Fin$ and $\mathcal{Q}(\ell_2)$ in the class of Boolean algebras and C^*-algebras of cardinality at most coninuum, respectivelly, Chapter 5 proves the analog for the Calkin algebra of the classical result of Brech and Koszmider saiyng that in the Cohen model the Boolean algebra $\mathcal{P}(\omega)/Fin$ is not universal in the class of Boolean of cardinality continuum in the sense that $\ell_\infty(\ell_\infty/c_0)$ does not embed in ℓ_∞/c_0 as a Banach space. More precisely, the main result of Chapter 5 states that in the Cohen model, there is no C^*-embedding of $\mathcal{Q}(\mathcal{Q}(\ell_2) \otimes \mathcal{K}(\ell_2))$ into the Calkin algebra $\mathcal{Q}(\ell_2)$. The proof of this non-commutative analogue of the classical result is considerably more intricate and requires new ideas not present in the commutative case.

The Thesis is well and carefully written, especially when it comes to its most important parts, proofs of the main results. I think that It could serve also as important reference to this area of mathematics if the author improves and unifies the stile of referencing especially in the introductory parts to the chapters. For example, in order to make this thesis an important reference to the field, when mentioning classical results that led to the desired developments in a particular area, original rather than secondary references should be given. (Of course, mentioning secondary references is also desirable.) To make my remark about uniformity of referencing clear, I take as example page 2 of the Introduction. In the three themes Kaplansky conjecture, biorthogonal systems, and existence of outer automorphisms of the Calkin algebra the

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referencing is not uniform. In the case of Calkin algebra the original references with names of authors spelled for both sides of the independence. In other two cases this is missing. For example, in the case of Kaplansky's conjecture the reference to papers of Dales and Esterle on one side and the work of Solovay and Woodin on the other side should be there. In the case of biorthogonal systems the reference to results of Kunen and Shelah on one side of the independence is missing and the name of the author of the paper supplying the other side of the independence is missing as well.. These are of course minor points and just given the current version of the Thesis as is, I am happy to very strongly support the motion to confer the degree of doctor of mathematical sciences to Damian Glodkowski.

Stevo Todorčević

Professor and Canada Research Chair

Bahen Center, 40 St. George St., Room 6290, Toronto, ON M5S2E4 Canada
Tel: +1 416 978-3323 • Fax: +1 416 978-4107 • www.math.utoronto.ca