Thermodynamic formalism and multifractal analysis for matrix cocycles and solenoids

Reza Mohammadpour Bejargafsheh

Supervisor: *Dr. hab. Michał Rams*

Instytut Matematyczny Polskiej Akademii Nauk

To my parents, Rahmat and Mehrangiz, and my brother, Arman, for their love, support and encouragement.

Oświadczam, że niniejsza rozprawa została napisana przeze mnie samodzielnie.

Reza Mohammadpour
Bejargafsheh

....
(data i podpis)

Niniejsza rozprawa jest gotowa do oceny przez recenzentów.

dr hab. Michał Rams

....
(data i podpis)

Contents

Acknowledgements

The completion of this thesis has been a long journey, step by step accompanied with resentment and hardships as well as enthusiasm and encouragement. I could not have finished it without the support of my family, supervisor, co-advisor, and friends.

First of all, I would like to express my deep gratitude to my supervisor, Michał Rams, and my co-advisor, Feliks Przytycki, for their guidance and time. In particular, I thank Michał for his careful reading of an earlier version of my papers and many helpful suggestions. I was also given a certain mathematical freedom to fully explore my own interests, knowing that I could always count on them. In addition to being my co-authors, they have given me knowledge and inspiration. I have also enjoyed many pleasant times in their company outside of work. During my PhD, I had the good fortune to attend some conferences and I thank Michał for allowing me to use his grant to attend them.

Secondly, I would like to thank the many friends that I made in Poland, in particular Jacopo and Adam for their company and all the memories.

A significant part of my life over the last four years was spent at IM PAN. I would like to thank IM PAN and its employees for the great hospitality and all the help. The atmosphere is inspiring for research and at the same time very friendly.

Thank you to all the people who helped me throughout this process.

Declarations

I declare that the materials in chapter 3 were obtained by myself, and based on the papers [M] which appears in the journal *Topological Methods in Nonlinear Analysis* and [Mo] which is available as a preprint on the arXiv.

The results given in chapter 4 of this thesis were obtained in collaboration with Feliks Przytycki and Michał Rams, and appear in the paper [MPR].

Abstract

We derive results in ergodic optimization, multifractal formalism and fractal geometry.

We prove that the restricted variational principle holds for generic matrix cocycles over subshifts of finite type, i.e,

$$
\begin{aligned}
h_{top}(E(\vec{\alpha})) &= \inf\{P_{\vec{\Phi_{\mathcal{A}}}}(\vec{q}) - \vec{\alpha}.\vec{q} : \vec{q} \in \mathbb{R}^d\} \\
&= \sup\{h_\mu(T) : \mu \in \mathcal{M}(X,T) \text{ with } \chi(\mu, \vec{\Phi}_{\mathcal{A}}) = \vec{\alpha}\},
\end{aligned}
$$

where $E(\vec{\alpha}) = \{x \in X;\ \lim_{n\to\infty} \frac{1}{n} \log \|(\mathcal{A}^{\wedge i})^n(x)\| = \alpha_i\}$.

We also show that for such cocycles over subshifts of finite type, the Lyapunov spectrum is equal to the closure of the set where the entropy spectrum is positive.

We consider a topological dynamical system, and define a subadditive potential Φ. We prove that for $t \to \infty$ any accumulation point of a family of equilibrium states of $t\Phi$ is a maximizing measure. We show that the Lyapunov exponent and entropy of equilibrium states for $t\Phi$ converge in the limit $t \to \infty$ to the maximum Lyapunov exponent and entropy of maximizing measures. We use the latter result to show the continuity of entropy spectrum at the boundary of Lyapunov spectrum for generic matrix cocycles.

We extend the continuity result of the lower joint spectral radius that was proven for locally constant cocycles by Bochi-Morris [BM] to derivative cocycles under an assumption that they admit a dominated splitting of index 1.

In the matrix cocycle case, we prove that the maximal Lyapunov exponent can be approximated by Lyapunov exponents of periodic trajectories under certain shadowing assumptions. Our approach differs considerably from the approach of Kalinin [Ka], who proved a similar result.

We also study a class of solenoidal expanding attractors Λ for which the contraction is not conformal. Under an assumption of transversality and assumptions on Lyapunov exponents for an appropriate Gibbs measure (stable Sinai-Ruelle-Bowen measure) imposing thinness, assuming also there is an invariant $C^{1+\alpha}$ strong stable foliation, we prove that Hausdorff dimension $\dim_H(\Lambda \cap W^s)$ is the same quantity t_0 for all W^s and else $\dim_H(\Lambda) = t_0 + 1$.

keywords:

zero temperature limits, maximal Lyapunov exponent, thermodynamic formalism, subadditive potentials, Lyapunov spectrum, matrix cocycles, domination, topological entropy, solenoid attractor, Hausdorff dimension.

Streszczenie

W pracy zajmuję się optymizacją ergodyczną, formalizmem multifraktalnym, i geometrią fraktalną.

Dowodzę dla typowych kocykli macierzowych nad subprzesunięciami skończonego typu tak zwaną ograniczoną zasadą wariacyjną (restricted variational principle), to znaczy

$$\begin{aligned} h_{top}(E(\vec{\alpha})) &= \inf\{P_{\vec{\Phi_{\mathcal{A}}}}(\vec{q}) - \vec{\alpha}.\vec{q} : \vec{q} \in \mathbb{R}^d\} \\ &= \sup\{h_\mu(T) : \mu \in \mathcal{M}(X,T) \text{ with } \chi(\mu, \vec{\Phi}_{\mathcal{A}}) = \vec{\alpha}\}, \end{aligned}$$

gdzie $E(\vec{\alpha}) = \{x \in X;\ \lim_{n\to\infty} \frac{1}{n} \log \|(\mathcal{A}^{\wedge i})^n(x)\| = \alpha_i\}$. Pokazuję również, że dla takich kocykli widmo wykładników Lapunowa jest dodatnie na gęstym podzbiorze.

Dla subaddytywnego potencjału Φ zadanego na topologicznym układzie dynamicznym pokazuję, że dowolna *słaba granica stanów równowagi dla $t\Phi$ przy $t \to 0$ jest miarą maksymalizującą wykładnik Lapunowa; zachodzi również zbieżność wykładnika Lapunowa i entropii. Ten wynik używam następnie do pokazania ciągłości entropii na granicy widma Lapunowa dla typowych kocykli macierzowych.

Ciągłość tak zwanego "lower joint spectral radius" została pokazana dla lokalnie stałych kocykli macierzowych przez Bochi'ego i Morrisa w [BM], rozszerzam ten wynik do kocykli z rozbiciem zdominowanym indeksu 1.

W klasie kocykli macierzowych dowodzę, że przy pewnych założeniach o aproksymacji trajektoriami okresowymi, maksymalny wykładnik Lapunowa przybliża się wykładnikami Lapunowa trajektorii okresowych. Podobny wynik został uzyskany innymi metodami przez Kalinina w [Ka].

W pracy badam również klasę solenoidalnych rozciągających atraktorów Λ z niekonforemną kontrakcją w kierunku stabilnym. Zakładając transwersalność, warunki na wykładniki Lapunowa pewnej miary Gibbsa (stabilnej miary Sinai'a-Ruelle'a-Bowena) implikujące "cienkość" atraktora, jak również istnienie $1+$hölderowskiej foliacji w kierunku silnie stabilnym, dowodzę, że wymiar Hausdorffa przecięcia atraktora z każdym liściem foliacji stabilnej przyjmuje tę samą wartość t_0. Dla całego solenoidu mamy $\dim_H(\Lambda) = t_0 + 1$.

Słowa kluczowe:

granica w zerowej temperaturze, maksymalny wykładnik Lapunowa, formalizm termodynamiczny, potencjały subaddytywne, widmo Lapunowa, kocykle macierzowe, warunek dominacji, entropia topologiczna, solenoid, wymiar Hausdorffa.

Chapter 1

Introduction

1.1 Motivation

The goal of this thesis is to present results on the subadditive thermodynamic formalism and their applications in different areas of dynamical systems. In particular, I will investigate

- restricted variational principle for matrix cocycles,
- continuity of the spectrum on the boundary (zero temperature limits),
- continuity properties of the equilibrium states,
- ...

Moreover, I will investigate the solenoids, a class of dynamical systems where the thermodynamic formalism will be used to obtain results on the geometric properties of the attractor. The thermodynamic formalism alone will not be enough, I will also use the smooth dynamical systems theory, in particular the properties of hyperbolic expanding attractors and the holonomy functions defined by projections along the one-dimensional unstable leaves.

1.1.1 Structure of thesis

The first chapter is the detailed introduction and presentation of results, basic definitions and theorems relevant to the work are given in chapter 2 and results are presented in chapters 3 to 4.

1.2 Presentation of results

In this thesis (X, T) denotes a *topological dynamical system* (TDS), that is, X is a compact metric space that is endowed by the metric d and $T : X \to X$ is a continuous map.

We denote by $\mathcal{M}(X,T)$ the space of all T-invariant Borel probability measures on X. This space is a nonempty convex set and is compact with respect to the weak-$*$ topology. Also let $\mathcal{E}(X,T) \subset \mathcal{M}(X,T)$ be the subset formed by ergodic measures, which are exactly the extremal points of $\mathcal{M}(X,T)$.

Let $f : X \to \mathbb{R}$ be a continuous function. We denote $S_n f(x) := \sum_{k=0}^{n-1} f(T^k(x))$ and call this a *Birkhoff sum* and we call $\lim_{n\to\infty} \frac{1}{n} S_n f(x)$ a *Birkhoff average.*

By Birkhoff ergodic theorem, for every $\mu \in \mathcal{M}(X,T)$ and μ-almost every $x \in X$, the Birkhoff average is well-defined. The infimum and the supremum of the Birkhoff average over $x \in X$ will be denoted by $\alpha(f)$ and $\beta(f)$, respectively; we call these numbers the *minimal and maximal ergodic averages of f.*

We say that $\Phi := \{\log \phi_n\}_{n=1}^{\infty}$ is a *subadditive potential* if each ϕ_n is a continuous positive-valued function on X such that

$$0 < \phi_{n+m}(x) \leq \phi_n(x)\phi_m(T^n(x)) \ \ \forall x \in X, m, n \in \mathbb{N}.$$

Furthermore, $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ is said to be an *almost additive potential* if there exists a constant $C \geq 1$ such that for any $m, n \in \mathbb{N}$, $x \in X$, we have

$$C^{-1}\phi_n(x)\phi_m(T^n)(x) \leq \phi_{n+m}(x) \leq C\phi_n(x)\phi_m(T^n(x)).$$

We also say that $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ is an *additive potential* if

$$\phi_{n+m}(x) = \phi_n(x)\phi_m(T^n(x)) \ \ \forall x \in X, m, n \in \mathbb{N};$$

in this case, $\phi_n(x) = e^{S_n \log \phi_1(x)}$.

We denote by $P_\Phi(t)$ the topological pressure for a potential $t\Phi$. We will give the definition in the next chapter.

As we mentioned above, the Birkhoff average does not exist for all points. So, one may ask about the size of the set of points

$$E_f(\alpha) = \{x \in X : \frac{1}{n} S_n f(x) \to \alpha \ \text{ as } \ n \to \infty\},$$

which we call *α-level set of Birkhoff spectrum*, for a given value α from the set

$$L = \{\alpha \in \mathbb{R} : \exists x \in X \ \text{ and} \lim_{n\to\infty} \frac{1}{n} S_n f(x) = \alpha\},$$

which we call *Birkhoff spectrum.*

That size is usually calculated in terms of topological entropy. Let $Z \subset X$, we denote by $h_{top}(T, Z)$ topological entropy of T restricted to Z or, simply, the topological entropy of Z, denote $h_{top}(Z)$, when there is no confusion about T. In particular we write $h_{top}(T)$ for $h_{top}(T, X)$.

We investigate the end points of Birkhoff spectrum, i.e., $\alpha(f)$ and $\beta(f)$. Since $\alpha(f) = -\beta(-f)$, let us focus the discussion on the quantity β. It can also be characterized as

$$\beta(f) = \sup_{\mu \in \mathcal{M}(X,T)} \int f d\mu.$$

Compactness of $\mathcal{M}(X, T)$ implies the following *attainability property*: there exists at least one measure $\mu \in \mathcal{M}(X, T)$ for which $\beta(f) = \int f d\mu$; such measures will be called *maximizing measures*.

We study the behavior of the equilibrium measures (μ_t) for a potential $t\Phi$ when $t \to \infty$. In the thermodynamic interpretation of the parameter t, it is the *inverse temperature*. The limits $t \to \infty$ are called *zero temperature limits*, and the accumulation points of the measure (μ_t) as $t \to \infty$ are called *ground states*.

The topic of *ergodic optimization of Birkhoff averages or Lyapunov exponents* revolves around realizing invariant measures which maximize the Lyapunov exponents. Zero temperature limits laws are also related to ergodic optimization, because for $t \to \infty$ any accumulation point of the equilibrium measures (μ_t) will be a maximizing measure Φ. We refer the reader to [BG] and [J].

The behavior of the equilibrium measure (μ_t) as $t \to \infty$ has also been analyzed. In particular, the continuities of zero temperature limit $(\mu_t)_{t\to\infty}$ in the sense,

$$\int f d\mu = \lim_{t\to\infty} \int f d\mu_t, \tag{1.2.1}$$

and

$$h_\mu(T) = \lim_{t\to\infty} h_{\mu_t}(T), \tag{1.2.2}$$

have been investigated by many authors [DUZ], [IY], [J], [JMU], [M1], [WZ], [Z].

In the non-compact space setting, (1.2.1) and (1.2.2) were proved by Jenkinson, Mauldin and Urbański [JMU], and Morris [M1] on the additive potential $\psi : X \to \mathbb{R}$. Moreover, this kind of result is known for almost subadditive potentials by Zhao [Z] under the specification property, upper semi-continuity of entropy and finite topological entropy assumptions.

Note that even though we know the existence of an accumulation point for the sequence (μ_t), this does not imply that the $\lim_{t\to\infty} \mu_t$ exists. In fact, Chazottes and Hochman [CH] constructed an example on compact sub-shifts of finite type and Hölder potentials, where there is no convergence. For more information about zero temperature limits see [J].

It is well known (see, e.g. [Ol], [Feng1], [FFW]) when (X,T) is a transitive subshift of finite type and f is an additive potential, then

$$E_f(\alpha) \neq \emptyset \Leftrightarrow \Omega := \{\int f d\mu : \mu \in \mathcal{M}(X,T)\},$$

and

$$\begin{aligned} h_{top}(E_f(\alpha)) &= \inf_{t\in\mathbb{R}}\{P_f(t) - \alpha t : t \in \mathbb{R}\} \\ &= \sup\{h_\mu(T) : \mu \in \mathcal{M}(X,T) \text{ with } \int f d\mu = \alpha\} \ \forall \alpha \in \Omega. \qquad (1.2.3) \end{aligned}$$

In the almost additive potentials case, (1.2.3) was proven by Feng and Huang [FH] under certain assumptions. In the subadditve potentials case, Feng and Huang [FH] proved a similar result for $t > 0$ under the upper semi continuity entropy assumption.

The natural example of subadditive potentials is matrix cocycles. More precisely, given a measurable map $\mathcal{A} : X \to GL(k, \mathbb{R})$ taking values into the space $k \times k$ invertible matrices. We consider the products

$$\mathcal{A}^n(x) = \mathcal{A}(T^{n-1}(x)) \dots \mathcal{A}(T(x))\mathcal{A}(x).$$

The pair $(T, \mathcal{A})$ is called a *linear cocycle*. It induces a skew-product dynamics F on $X \times \mathbb{R}^k$ by $(x, v) \mapsto X \times \mathbb{R}^k$, whose n-th iterate is therefore

$$(x, v) \mapsto (T^n(x), \mathcal{A}^n(x)v).$$

If T is invertible then so is F. Moreover, $F^{-n}(x) = (T^{-n}(x), \mathcal{A}^{-n}(x)v)$ for each $n \geq 1$, where

$$\mathcal{A}^{-n}(x) := \mathcal{A}(T^{-n}(x))^{-1}\mathcal{A}(T^{-n+1}(x))^{-1}...\mathcal{A}(T^{-1}(x))^{-1}.$$

More generally, we could replace $X \times \mathbb{R}^k$ by any vector bundle over X and then consider bundle endomorphisms that fiber over $T : X \to X$.

A simple class of linear cocycles is *locally constant cocycles* which is defined as follows. Assume that $X = \{1, ..., q\}^{\mathbb{Z}}$ is a symbolic space. Suppose that $T : X \to X$ is a shift map, i.e. $T(x_l)_l = (x_{l+1})_l$. Given a finite set of matrices $\mathcal{A} = \{A_1, \dots, A_q\} \subset GL(k, \mathbb{R})$, we define the function $\mathcal{A} : X \to GL(k, \mathbb{R})$ by $\mathcal{A}(x) = A_{x_0}$. In this case, we say that$(T, \mathcal{A})$ is a locally constant cocycle.

By Kingman's subadditive ergodic theorem, for any $\mu \in \mathcal{M}(X,T)$ and μ almost every $x \in X$ such that $\log^+ \|\mathcal{A}\| \in L^1(\mu)$, the following limit, called the *top Lyapunov exponent* at x, exists:

$$\chi(x,\mathcal{A}) := \lim_{n\to\infty}\frac{1}{n}\log\|\mathcal{A}^n(x)\|, \tag{1.2.4}$$

where $\|\mathcal{A}\|$ the Euclidean operator norm of a matrix $\mathcal{A}$ (i.e. the largest singular value of $\mathcal{A}$), that is subadditive i.e.,

$$0 < \|\mathcal{A}^{n+m}(x)\| \leq \|\mathcal{A}^n(x)\|\|\mathcal{A}^n(T^m(x))\| \ \ \forall x \in X, m, n \in \mathbb{N}.$$

Let us denote $\chi(\mu,\mathcal{A}) = \int \chi(.,\mathcal{A})d\mu$. If the measure μ is ergodic then $\chi(x,\mathcal{A}) = \chi(\mu,\mathcal{A})$ for μ-almost every $x \in X$.

Similarly to what we did for the Birkhoff average, we can either minimize or maximize number (1.2.4); the corresponding quantities will be denoted by $\alpha(\mathcal{A})$ and $\beta(\mathcal{A})$. However, this time the maximization and the minimization problems are totally different. While $\beta(\mathcal{A})$ is always attained by at least one measure (which will be called a *Lyapunov maximizing measure*, we denote by $\mathcal{M}_{\max}(\mathcal{A})$ the set of such measures), that is not necessarily the case for $\alpha(\mathcal{A})$. In fact, in the locally constant cocycles case, Bochi and Morris [BM] investigated the continuity properties of the minimal Lyapunov exponent. They showed that $\alpha(\mathcal{A})$ is Lipschitz continuous at $\mathcal{A}$ under $1-$domination assumption. Breuillard and Sert [BS] extended the Bochi and Morris's result to the joint spectrum under domination condition. In this case the $\chi(\mu,\mathcal{A})$ depends continuously on the measure μ.

Feng [Feng1] proved (1.2.3) for continuous positive matrix-valued functions on the one side shift. He (see [F], [FH])also proved that the first part (1.2.3) for locally constant cocycles under the irreducibility assumption.

The linear cocycles generated by a diffemorphism map $T : X \to X$ on a closed Riemannian manifold X and a family of maps $\mathcal{A}(x) := D_xT : T_xX \to T_{T(x)}X$ are called *derivative cocycles*. Moreover, when $T : X \to X$ is an Anosov diffemophism (or expanding map), Bowen [B] showed that there exists a symbolic coding of T by a subshift of finite type. From such a coding, the derivative cocycle of a uniformly hyperbolic map can effectively be regarded as a linear cocycle over a subshift of finite type.

The main objects of interest in this thesis are linear cocycles $\mathcal{A}$ over two-sided subshifts of finite type (Σ, T) generated by $GL(k,\mathbb{R})$-valued functions $\mathcal{A}$ on Σ. In particular, we study the thermodynamic formalism of such cocycles. In general, we know much more about locally constant cocycles that about the more general derivative cocycles, but here are some of the results known in the derivative cocycles situation.

We denote by $\mathcal{L}$ the set of admissible words Σ. We define for $\mathcal{A} : \Sigma \to GL(k,\mathbb{R})$ and $I \in \mathcal{L}$

$$\|\mathcal{A}(I)\| := \max_{x\in[I]}\|\mathcal{A}^{|I|}(x)\|. \tag{1.2.5}$$

We define a positive continuous function $\{\varphi_{\mathcal{A},n}\}_{n\in\mathbb{N}}$ on Σ such that

$$\varphi_{\mathcal{A},n}(x) := \|\mathcal{A}^n(x)\|.$$

We denote by $\Phi_{\mathcal{A}}$ the subbadditive potential $\{\log \varphi_{\mathcal{A},n}\}_{n=1}^{\infty}$.

We say that $\mathcal{A}$ is *quasi-multiplicative* if there exist $C > 0$ and $m \in \mathbb{N}$ such that for every $I, J \in \mathcal{L}$, there exists $K \in \mathcal{L}$ with $|K| \leq m$ such that $IKJ \in \mathcal{L}$ and

$$\|\mathcal{A}(IKJ)\| \geq C\|\mathcal{A}(I)\|\|\mathcal{A}(J)\|.$$

We always assume that $T : \Sigma \to \Sigma$ is a topologically mixing subshift of finite type. We denote by $H^r(\Sigma, GL(k, \mathbb{R}))$ the space of all $r-$Hölder continuous functions. We also denote by $H_b^r(\Sigma, GL(k, \mathbb{R}))$ the space of all $r-$Hölder continuous and fiber bunched functions, which says that the cocycles are nearly conformal. We define the *typical cocycles* among $H^r(\Sigma, GL(k, \mathbb{R}))$. That is

$$\mathcal{W} := \{\mathcal{A} \in H_b^r(\Sigma, GL(k, \mathbb{R})) : \mathcal{A} \text{ is pinching and twisting}\}.$$

We denote $E(\alpha) = E_\Phi(\alpha)$ when there is no confusion about Φ.

The results of this thesis are as follows:

Theorem 1.2.1. *Let $\mathcal{A} \in \mathcal{W}$. Then,*

$$L = \overline{\{\alpha,\ \ h_{top}(E(\alpha)) > 0\}}.$$

Furthermore, $\alpha \mapsto h_{top}(E(\alpha))$ is concave for $\alpha \in \mathring{L}$.

We also prove (1.2.3) for generic cocycles. Park [P] showed that every $\mathcal{A} \in \mathcal{W}$ is quasi-multiplicative. That implies that $P_{\Phi_{\mathcal{A}}}(q)$ is convex for $q \in \mathbb{R}$.

Theorem 1.2.2. *Assume that $T : \Sigma \to \Sigma$ is a topologically mixing subshift of finite type. Suppose that $\mathcal{A} : \Sigma \to GL(k, \mathbb{R})$ belongs to typical functions $\mathcal{W}$. Then,*

$$\begin{aligned} h_{top}(E(\alpha)) &= \sup\{h_\mu(T) : \mu \in \mathcal{M}(\Sigma, T), \chi(\mu, \mathcal{A}) = \alpha\} \\ &= \inf\{P_{\Phi_{\mathcal{A}}}(q) - \alpha.q : q \in \mathbb{R}\} \ \ \forall \alpha \in \Omega. \end{aligned}$$

We also extend the zero temperature limit and the continuity results for subadditive potentials.

Theorem 1.2.3. *Let (X, T) be a TDS such that the entropy map $\mu \mapsto h_\mu(T)$ is upper semi-continuous and topological entropy $h_{top}(T) < \infty$. Suppose that $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ is a subadditive potential on the compact metric on X. Then a family of equilibrium measures (μ_t) for potentials $t\Phi$, where $t > 0$, has a weak* accumulation point as $t \to \infty$. Any such accumulation point μ is a Lyapunov maximizing measure for Φ. Moreover,* *(i)* $\chi(\mu, \Phi) = \lim_{t\to\infty} \chi(\mu_t, \Phi)$,

(ii) $h_\mu(T) = \lim_{t\to\infty} h_{\mu_t}(T) = \max\{h_\nu(T), \nu \in \mathcal{M}_{\max}(\Phi)\}$.

Furthermore, $\beta(\Phi)$ can be approximated by Lyapunov exponents of equilibrium measures of a subadditive potential $t\Phi$.

Theorem 1.2.4. *Suppose $\mathcal{A}_l, \mathcal{A} \in \mathcal{W}$ with $\mathcal{A}_l \to A$, and $t_l, t \in \mathbb{R}_+$ such that $t_l \to t$. Assume $\alpha_{t_l} = P'_{\Phi_{\mathcal{A}_l}}(t_l)$ and $\alpha_t = P'_{\Phi_{\mathcal{A}}}(t)$. Then,*

$$\lim_{l\to\infty} h_{top}(E(\alpha_{t_l})) = h_{top}(E(\alpha_t)).$$

Moreover,

$$h_{top}(E(\alpha_t)) \to h_{top}(E(\beta(\Phi_{\mathcal{A}})) \text{ when } t \to \infty.$$

We also investigate the continuity of minimal Lyapunov exponents for general cocycles. We prove the continuity of the minimal Lyapunov exponent under a cone condition. Moreover, our result implies the continuity of the minimal Lyapunov exponent under 1-domination assumption.

Theorem 1.2.5. *Let $\mathcal{A}_n, \mathcal{A} \in H_b^r(\Sigma, GL(k,\mathbb{R}))$. Assume that $\mathcal{A}_n$ and $\mathcal{A}$ satisfy $1-$domination. Then, $\alpha(\mathcal{A}_n) \to \alpha(\mathcal{A})$, when $\mathcal{A}_n \to \mathcal{A}$.*

We define the singular value function $\varphi^s : GL(k,\mathbb{R}) \to [0,\infty)$ with the parameter $0 \le s \le k$ as follows.

$$\varphi^s(\mathcal{A}) = \sigma_1(\mathcal{A})...\sigma_m(\mathcal{A})\sigma_{m+1}^{s-m}(\mathcal{A}),$$

where $m = \lfloor s \rfloor$ and σ_i is the i th singular value. We make the convention $0^0 = 1$. For completeness, if $s > k$, the we also define

$$\varphi^s(\mathcal{A}) = (\det(\mathcal{A}))^{\frac{s}{d}}.$$

It is well known that φ^s is *submultiplicative* for all $s \ge 0$. That means, for any $A, B \in GL(k,\mathbb{R})$

$$\varphi^s(AB) \le \varphi^s(A)\varphi^s(B).$$

The function $(s, A) \mapsto \varphi^s(A)$ is continuous in both A and s where $A \in GL(k,\mathbb{R})$.

We define $\tilde{\varphi^s_{\mathcal{A}}}$ on $\mathcal{L}$ as follows, for any $n \in \mathbb{N}$ and $I \in \mathcal{L}(n)$,

$$\tilde{\varphi^s_{\mathcal{A}}}(I) := \max_{x\in[I]} \varphi^s(\mathcal{A}^n(x)).$$

Note that this definition is similar to how we define $\|\mathcal{A}(I)\|$ in (1.2.5). From the submultiplicativity of φ^s, it follows that $\varphi^s_{\mathcal{A}}$ is also submultiplicative. We denote by $\tilde{\Phi}_{\mathcal{A}} := \{\log \varphi^s(\mathcal{A}^n)\}$.

Feng and Shmerkin [FS] showed the continuity of the topological pressure for locally constant cocycles. Moreover, this kind of result is known for typical cocycles by Park. Recently, Cao, Pesin, and Zhao [CPZ] showed that the map $(s, \mathcal{A}) \mapsto P_{\tilde{\Phi}_{\mathcal{A}}}(s)$ is continuous on $[0, \infty) \times H^r(\Sigma, GL(k, \mathbb{R}))$, and Theorem 1.2.6 is implied by their result. However, the methods of proof are different.

We show that one can prove the continuity of the topological pressure for $H_b^r(\Sigma, GL(2, \mathbb{R}))$ without assumption pinching, and twisting.

Theorem 1.2.6. *The map $(s, \mathcal{A}) \mapsto P_{\tilde{\Phi}_{\mathcal{A}}}(s)$ is continuous on $[0, \infty) \times H_b^r(\Sigma, GL(2, \mathbb{R}))$.*

We considered the linear cocycles as abstract objects but now we will investigate a natural example of geometric appearance of matrix cocycles: solenoid.

Let $M = S^1 \times \mathbb{D}$ be the solid torus, where $\mathbb{D} = \{v \in \mathbb{R}^2 ||v| < 1\}$ carries the product distance $d = d_1 \times d_2$ and suppose $f : M \to M$ such that

$$(x, y, z) \mapsto (\eta(x, y, z) \mod 2\pi, \lambda(x, y, z) + u(x), \nu(x, y, z) + v(x)) \tag{1.2.6}$$

is a $C^{1+\alpha}$ invective map, where $\lambda(x, 0, 0) = \nu(x, 0, 0) = 0$.

Bothe [Bot] was the first who obtained results on the dimension of the attractor of a thin linear solenoid under transversality condition, which we will introduce in the last chapter. He also proved that this transversality condition holds generically when the contractions are strong enough. Simon [Simon] use Bothe's result to show that the Hausdroff dimension of all stable slices are equal. Barriera, Pesin and Schemeling [BPS] established a dimension product structure of invariant measures in the course of proving the following conjecture.

Conjecture. *The fractal dimension of a hyperbolic set is (at least generically or under mild hypotheses) the sum of those of its stable and unstable slices, where fractal can mean either Hausdorff or upper box dimension.*

There are difficulties due to possible low regularity of the holonomies, indeed, Schmeling [Sch] found that while the solenoids often lack regular holonomies, under natural assumptions there exist bounds on the size of the set of non-Lipschitz points for the holonomy map. We will provide the details of this result later.. Hasselblat and Schmeling [HS] proved the conjecture for a class of thin linear solenoids. We prove the conjecture for a class of thin nonlinear solenoids of map (1.2.6). Precisely,

Theorem 1.2.7. *Under transversality and $\chi(\mu_{t_0}, \nu') < \chi(\mu_{t_0}, \lambda') < -\chi(\mu_{t_0}, \eta')$ assumptions, we show that Hausdorff dimension of the conditional measures on $W^s \cap S^1$ of the geometric equilibrium measure μ_t for f^{-1} and the potential $t_0 \log \lambda'$ (or stable slices), where λ is the weaker contraction rate function, is t_0. Then, we show that the Hausdorff dimension of solenoid attractor is $1 + t_0$.*

Chapter 2

Preliminaries

2.1 Ergodic theory

We introduce some basic notions from dynamical systems and ergodic theory. Let the triple $(X, \mathcal{B}, \mu)$ denote the space X equipped with a σ-algebra $\mathcal{B}$ of measurable subsets of X and a probability measure μ. Let $T : X \to X$ be a transformation. Then, we say that (X, T) is a *dynamical system.* Given a point $x \in X$ we say that $\{x, Tx, T^2x, ...\}$ is the orbit of x under T. For a subset $A \subset X$ denote $T^{-1}(A) = \{x \in X : T(x) \in A\}$. We say that T is measurable if for all $A \in \mathcal{B}$, $T^{-1}(A) \in \mathcal{B}$. We say that a set $A \subset X$ is *forward invariant set* if $T(A) \subset A$. If T is an invertible, then we say that a set $A \subset X$ is *backward invariant set* if $T^{-1}(A) \subset A$. A set $A \subset X$ is *invariant set* if it is forward and backward invariant set. We say that T is measure preserving if $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$, and in this case we may also say that μ is T-invariant (or just invariant, whenever the choice of map is clear).

Assume that $(X, \mathcal{B}, \mu, T)$ is a measure preserving transformation. $T : X \to X$ is said to be *ergodic* if for any set $A \in \mathcal{B}$ which satisfies $T^{-1}(A) = A$ then either $\mu(A) = 0$ or $\mu(A) = 1$. That is equivalent, if $\psi \in L^1(\mu)$ is T-invariant, i.e. $\psi \circ T = \psi$ μ-a.e., then ψ is constant μ-a.e. . Although T can have many ergodic measures, distinct ergodic measures μ_1 and μ_2 are mutually singular, meaning that there exists $A \in \mathcal{B}$ for which $\mu_1(A) = \mu_2(X \backslash A) = 1$. Given an ergodic transformation, we can deduce various statistical properties of T. The most well-known of these is the Birkhoff ergodic theorem, which connects the average of a potential f along the orbit of a μ-typical point with the space average of f.

Theorem 2.1.1 (Birkhoff Ergodic Theorem)**.** *Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic measure preserving transformation such that $\mu(X) = 1$. Let $f \in L^1(\mu)$.*

Then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x)) = \int f d\mu$$

for μ a.e. $x \in X$.

Theorem 2.1.2 ([King, Kingman's subadditive theorem]). *Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be a measure-preserving transformation. Let $f_n : X \to [-\infty, \infty)$ be a subadditive sequence of measurable functions such that $f_1 \in L^1(\mu)$, i.e. $f_{n+m}(x) \leq f_n(x) f_m(T^n(x))$ for all $x \in X$, and $n, m \in \mathbb{N}$. Then, $\lim_{n\to\infty} \frac{f_n(x)}{n}$ converges $\mu-$almost every where to some invariant function $f : X \to [-\infty, \infty)$. Moreover, the positive part f^+ is integrable and*

$$\int f d\mu = \lim_{n\to\infty}\frac{1}{n}\int f_n d\mu = \inf_n \frac{1}{n}\int f_n d\mu \in [-\infty, \infty).$$

Let (X, τ, μ) be a Borel probability space, and $T : X \to X$ be a measure preserving transformation.

A *partition* of (X, τ, μ) is a subfamily of τ consisting of mutually disjoint elements whose union is X. We denote by α and β the countable partition of X.

Let $\alpha = \{A_i, i \geq 1\}$, where $A_i \in \tau$. We define

$$H_\mu(\alpha) = -\sum_{A\in\alpha} \mu(A)\log\mu(A)$$

to be the *entropy* of α (with the convention $0\log 0 = 0$).

We denote by $\alpha \vee \beta$ the joint partition $\{A \cap B \mid A \in \alpha, B \in \beta\}$.

Let $T^{-1}(\alpha) = \{T^{-1}(A) \mid A \in \alpha\}$. We define

$$h(\mu, \alpha) = \lim_{n\to\infty}\frac{1}{n}H_\mu(\bigvee_{j=0}^{n-1} T^{-1}(\alpha))$$

to be the entropy of T relative to α [1].

Then the metric entropy of μ is defined as

$$h_\mu(T) = \sup h(\mu, \alpha),$$

where the supremum is taken over all countable partitions α with $H_\mu(\alpha) < \infty$. By a well-known theorem of Kolmogorov and Sinai (e.g. [W1, Theorem 4.1.7]), $h_\mu(T) = h(\mu, \alpha)$ for any partition α such that $H_\mu(\alpha) < \infty$ and $\bigvee_{j=0}^{n-1} T^{-1}(\alpha) \to \mathcal{B}$ as $n \to \infty$.

[1] Limits exist by subadditivity.

Take into account that $\mu \mapsto h_\mu(T)$ is concave on $\mathcal{M}(X,T)$ (c.f [PU, Chapter 2]). We also have the following important theorem which gives an alternative characterization for the entropy when T is ergodic (see for instance the remark below Corollary 4.14.4 in [W1]).

Theorem 2.1.3 (Shannon-McMillan-Breiman Theorem)**.** *Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic measure-preserving transformation of a probability space and. let α be a finite partition of X. Let $B_n(x)$ denote the unique member of $\bigvee_{j=0}^{n-1} T^{-1}(\alpha)$ to which x belongs. Then,*

$$-\lim_{n\to\infty} \frac{1}{n} \log \mu(B_n(x)) = h_\mu(T)$$

for μ-almost every x.

2.1.1 Topological entropy

Assume that (X,d) is a topological dynamical systems. For any $n \in \mathbb{N}$ we define a new metric d_n on X as follows

$$d_n(x,y) = \max\{d(T^k(x), T^k(y)) : k = 0, ..., n-1\},$$

and for any $\varepsilon > 0$, one can define *Bowen ball* $B_n(x,\varepsilon)$ that is an open ball of radius $\varepsilon > 0$ in the metric d_n around x. That is,

$$B_n(x,\varepsilon) = \{y \in X : d_n(x,y) < \varepsilon\}.$$

Let $Y \subset X$ and assume that $Y \subset \bigcup_i B_{n_i}(x_i,\varepsilon)$ for some at most countable collection of Bowen balls $\mathcal{Y} = (B_{n_i}(x_i,\varepsilon))_i$. Consider $N(\mathcal{Y}) = \min_i n_i$. Let $s \geq 0$ and

$$S(Y,s,N,\varepsilon) = \inf \sum_i e^{-sn_i},$$

where the infinum is taken over all collections $\mathcal{Y} = \{B_{n_i}(x_i,\varepsilon)\}_i$ covering Y such that $n(\mathcal{Y}) \geq N$. The quantity $S(Y,s,N,\varepsilon)$ does not decrease with N, consequently

$$S(Y,s,\varepsilon) = \lim_{N\to\infty} S(Y,s,N,\varepsilon).$$

There is a critical value of the parameter s, which we denote by $h_{top}(T,Y,\varepsilon)$ such that

$$S(Y,s,\varepsilon) = \begin{cases} 0, & s > h_{top}(T,Y,\varepsilon), \\ \infty, & s < h_{top}(T,Y,\varepsilon). \end{cases}$$

Since $h_{top}(T,Y,\varepsilon)$ does not decrease with ε, the following limit exists,

$$h_{top}(T,Y) = \lim_{\varepsilon\to 0}(T,Y,\varepsilon).$$

We call $h_{top}(T,Y)$ the *topological entropy* of T restricted to Y or the topological entropy of Y (we denote $h_{top}(Y)$), as there is no confusion about T. We denote $h_{top}(X,T) = h_{top}(T)$. Various such definition has been given by Bowen [B1] and Pesin and Pitskel [PP].

2.2 Symbolic dynamic

We discuss symbolic dynamics, in particular topological Markov shifts which plays as an important model throughout this thesis.

Let $Q = (q_{ij})$ be a $k \times k$ with $q_{ij} \in \{0,1\}$. The one side subshift of finite type associated to the matrix Q is a left shift map $T : \Sigma_Q^+ \to \Sigma_Q^+$ meaning that, $T(x_n)_{n\in\mathbb{N}_0} = (x_{n+1})_{n\in\mathbb{N}_0}$, where Σ_Q^+ is a set of sequences

$$\Sigma_Q^+ := \{x = (x_i)_{i\in\mathbb{N}_0} : x_i \in \{1,...,k\} \text{ and } Q_{x_i,x_{i+1}} = 1 \text{ for all } i \in \mathbb{N}_0\},$$

Similarly, one defines two sided subshift of finite type $T : \Sigma_Q \to \Sigma_Q$, where

$$\Sigma_Q := \{x = (x_i)_{i\in\mathbb{Z}} : x_i \in \{1,...,k\} \text{ and } Q_{x_i,x_{i+1}} = 1 \text{ for all } i \in \mathbb{Z}\}.$$

When the matrix Q has entries all equal to 1 we say this is the *full shift*. For simplicity, we assume that $\Sigma_Q^+ = \Sigma^+$ and $\Sigma_Q = \Sigma$.

We say that $i_1...i_k$ is an *admissible word* if $Q_{i_n,i_{n+1}} = 1$ for all $1 \leq n \leq k-1$. We denote by $\mathcal{L}$ the set of collection of admissible words. We denote by $|I|$ the length of $I \in \mathcal{L}$. Denote by $\mathcal{L}(n)$ the *admissible words of length* n. That is, a word $i_0,..,i_{n-1}$ with $i_j \in \{1,...,k\}$ such that $Q_{x_i,x_{i+1}} = 1$. One can define *n-th level cylinder* $[I]$ as follows:

$$[I] = [i_0...i_{n-1}] := \{x \in \Sigma : x_i = i_j \ \forall\, 0 \leq j \leq n-1\},$$

for any $i_0...i_{n-1} \in \mathcal{L}(n)$.

Observe that the partition of Σ_Q (or Σ_Q^+) into first level cylinders is generating, for this reason the partition into first level cylinders is the partition canonically used in symbolic dynamics to calculate the metric entropy.

In the two-sided dynamics, we define the *local stable set*

$$W^s_{\text{loc}}(x) = \{(y_n)_{n\in\mathbb{Z}} : x_n = y_n \text{ for all } n \geq 0\}$$

and the *local unstable set*

$$W^u_{\text{loc}}(x) = \{(y_n)_{n\in\mathbb{Z}} : x_n = y_n \text{ for all } n < 0\}.$$

Furthermore, the global stable and unstable manifolds of x,

$$W^s(x) := \cup_{n=0}^{\infty} T^{-n}(W^s_{\text{loc}}(T^n(x))) \text{ and } W^u(x) := \cup_{n=0}^{\infty} T^{n}(W^s_{\text{loc}}(T^{-n}(x)))$$

are smoothly immersed submanifolds of X and they are characterized by

$$W^s(x) = \{y \in X : \lim_{n\to\infty} d(T^n(x), T^n(y)) = 0\},$$

$$W^u(x) = \{y \in X : \lim_{n\to-\infty} d(T^n(x), T^n(y)) = 0\}.$$

Definition 2.2.1. The matrix Q is called primitive when there exist n such that all the entries of Q^n are positive.

It is well known that a subshift of a finite type associated with a primitive matrix Q is *topologically mixing* T. That is, for every open nonempty $U, V \subset \Sigma$, there is N such that for every $n \geq N$, $T^n(U) \cap V \neq \emptyset$. We say that T is *topological transitive* if there is a point with dense orbit.

We will introduce two important classes of shift-invariant measures.

2.2.1 Bernoulli measures

Let (Σ, σ) be the full shift on the alphabet A, where $A = \{1, ..., k\}$. Let $\vec{p} = (p_1, p_2, \ldots, p_k)$ be a probability vector, that is, $p_i \geq 0$ with $\sum_{i=1}^k p_i = 1$.

By the Kolmogorov extension theorem, to define a Borel measure on Σ it is sufficient to define a measure on the cylinder sets. We define the measure $\mu_{\vec{p}}$ on the cylinder sets of σ by

$$\mu_{\vec{p}}([i_1, \ldots, i_s]) = p_{i_1} \ldots p_{i_s}$$

and say that $\mu_{\vec{p}}$ is a *Bernoulli measure* for $\vec{p}$. Then, $(\sigma, \Sigma, \mu_{\vec{p}})$ is an ergodic measure preserving system. Bernoulli measure can be defined both for one-sided and for two-sided full shift.

2.2.2 Gibbs measures

A probability measure μ on Σ, where is the one-sided symbolic dynamics, is said to be a *Gibbs state (measure)* for the continuous function $\phi : \Sigma \to \mathbb{R}$ (it is called *potential*) if there exist $P \in \mathbb{R}$ and $C \geq 1$ such that for all $n \geq 1$, and $I \in \mathcal{L}_n$, we have

$$C^{-1} \leq \frac{\mu([I])}{e^{S_n\phi(x) - nP}} \leq C,$$

for any $x \in I$. If in addition μ is T-invariant, we call μ an invariant Gibbs state (measure).

For systems like hyperbolic systems which there is a Markov coding for them, one can also define a Gibbs measure as above (see for more information [PU, Chapter 5]).

Gibbs measures were translated from statistical mechanics to the setting of dynamical systems by Ruelle and Sinai beginning with [Si1], providing a class of invariant measures whose properties were closely connected with the properties of the Gibbs potential.

2.3 Convex functions

We first give some notation and basic facts in convex analysis. For details, one is referred to [HL].

Let $x, y \in \mathbb{R}^n$, the line segment connecting x and y is the set $[x, y]$ formally given by

$$[x, y] = \{\beta x + (1-\beta)y \ \ \beta \in [0, 1]\}.$$

We say that a set $X \subset \mathbb{R}^n$ is convex when for any two points $x, y \in X$, the line segment $[x, y]$ also belongs to the set X, i.e., $\beta x + (1-\beta)y \in X$ for any $x, y \in X$ and $\beta \in (0, 1)$. Let C be a convex subset of $\mathbb{R}^n$. A point $x \in C$ is called an *extreme point* of C if whenever $x = \beta y + (1-\beta)z$ for some $y, z \in C$ and $0 < \beta < 1$, then $x = y = z$. The set of extreme points of C is denoted by $\text{ext}(C)$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is a convex if its domain $\text{dom}(f)$ is a convex set and for all $x,y \in \text{dom}(f)$ and $\beta \in (0, 1)$, the following relation holds

$$f(\beta x + (1-\beta)y)) \leq \beta f(x) + (1-\beta)f(y).$$

In other words, a function $f : \mathbb{R}^n \to \mathbb{R}$ is convex when for every segment $[x_1, x_2]$, as the vector $x_\beta = \beta x_1 + (1-\beta)x_2$ varies within the line segment $[x_1, x_2]$, the points $(x_\beta, f(x_\beta))$ on the graph $\{(x, f(x))|x \in \mathbb{R}^n\}$ lie below the segment connecting $(x_1, f(x_1))$ and $(x_2, f(x_2))$, as illustrated in Figure 1.

Let U be an open convex subset of $\mathbb{R}^n$ and f be a real continuous convex function on U. We say a vector $a \in \mathbb{R}^n$ is a *subgradient* of f at x if for all $z \in U$,

$$f(z) \geq f(x) + a^T(z - x),$$

where the right hand side is the scalar product.

For each $x \in \mathbb{R}^n$ set the *subdifferential* of f at the point x to be

$$\partial f(x) := \{a : a \text{ is a subgradient for } f \text{ at } x\}.$$

For $x \in U$, the subdifferential $\partial f(x)$ is always a nonempty convex compact set. Define $\partial^e f(x) := \text{ext}\{\partial f(x)\}$. In case $n = 1$, $\partial^e f(x) = \{f^{'}(x_-), f^{'}(x_+)\}$. We say that f is *differentiable* at x when $\partial^e f(x) = \{a\}$.

We define

$$\partial f(U) = \cup_{x \in U} \partial f(x) \ \text{ and } \ \partial^e f(U) = \cup_{x \in U} \partial^e f(x). \tag{2.3.1}$$

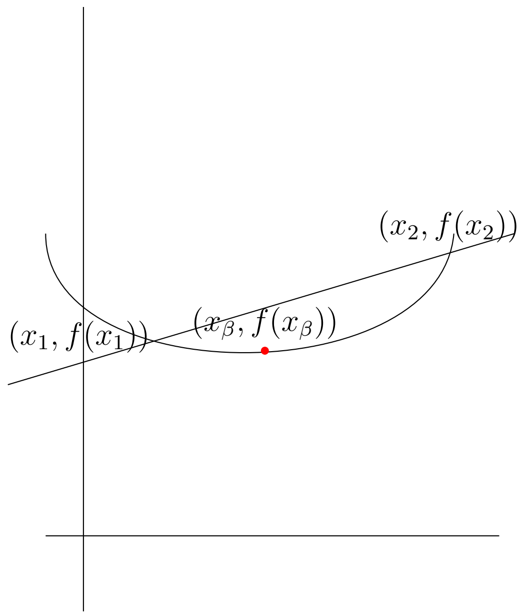


Figure 1: Convex line

In the case $n = 1$, Lebesgue's theorem for the differentiability of monotone functions said $\partial^e f$ is differentiable almost everywhere. The case $n = 2$ was proven by H. Busemann and W. Feller [BF]. The general case was settled by A. D. Alexandrov [A]. The following result is well known(cf. [[S], Theorem 7.9]).

Theorem 2.3.1. *Let f be a continuous function defined on an open interval that has a derivative at each point of $\mathbb{R}$ except on a countable set, and $f^{'} \leq 0$ a.e., then f is a nonincreasing function.*

2.4 Lyapunov exponents

Lyapunov exponents are named after Aleksandr Mikhailovich Lyapunov [lya], because of his fundamental work on the stability of solutions of differential equations in the late 19th century. Consider a quasi-linear differential equation

$$x^{'} = L(t)x + R(t, x), \tag{2.4.1}$$

where $L(t) : \mathbb{R}^d \to \mathbb{R}^d$ is linear and $R(t, x)$ is a perturbation of order bigger than 1:

$$\sup_t \frac{\|R(t, x)\|}{\|x\|} \to 0 \text{ as } x \to 0.$$

Let t_0 be fixed. The *Lyapunov exponent function* $v \mapsto \chi(v)$ is defined by

$$\chi(v) = \limsup_{t\to\infty} \frac{1}{t} \log \|\beta_v(t)\| \tag{2.4.2}$$

where β_v denotes the solution of the linear equation

$$x' = L(t)x \tag{2.4.3}$$

with initial condition $\beta_v(t_0) = v$. It does not depend on the choice of t_0.

It is clear that the solution $\beta_0(t) \equiv 0$ of the linear equation (2.4.3) is exponentially stable if $\chi < 0$. The stability theorem of Lyapunov states that the zero solution remains exponentially stable for the non-linear equation (2.4.1), under an additional condition called *Lyapunov regularity*. See for more information of Barreira, Pesin [BP07], which contains a detailed presentation of this topic.

Furstenberg and Kesten [FK1] proved in 1960 that the limit in (2.4.2) exists for almost every x, relative to any probability measure invariant under the flow. A few years later, in 1968, Oseledets [Ose] proved that Lyapunov regularity also holds for almost every point. These two results brought the subject of Lyapunov exponents firmly to the camp of ergodic theory, where it has prospered since. To give their precise statements, we need the notion of linear cocycle.

The work of Furstenberg, Ledrappier, Guivarc'h, Raugi, Gol'dsheid, Margulis, Mañè, Viana, Bonatti, Avila, Bochi and other mathematicians, built the study of Lyapunov characteristic exponents into a very active research field in its own right, and one with an unusually vast array of interactions with other areas of Mathematics and Physics, such as stochastic processes (random matrices and, more generally, random walks on groups), spectral theory (Schrödinger-type operators) and smooth dynamics (non-uniform hyperbolicity), to mention just a few.

2.4.1 Theorem of Oseledets

This is a refinement of Furstenberg and Kesten's theorem [FK1] in that the conclusion is formulated in terms of the norms of the images $\|\mathcal{A}^n(x)v\|$, for every non-zero $v \in \mathbb{R}^d$, rather than the norm $\|\mathcal{A}^n(x)\|$ of the matrix itself. That is, while Furstenberg and Kesten's theorem is concerned with the matrices $\mathcal{A}^n(x)$, the next statement is about their individual column vectors.

Theorem 2.4.1 ([Ose, Oseledets]). *Assume that $\log^+ \|\mathcal{A}\|$ is integrable with respect to μ. Then at μ-almost every $x \in X$ there exist an integer $k(x) \geq 1$, a flag $\mathbb{R}^d = V_x^1 > \ldots > V_x^{k(x)} > \{0\}$, and real numbers $\chi_1(x,\mathcal{A}), \ldots, \chi_{k(x)}(x,\mathcal{A})$ such that for any $i = 1, \ldots, k(x)$,*

1) the functions $x \mapsto k(x), \chi_i(x,\mathcal{A}), V_x^i$ are measurable;

2) $k(x) = k(T(x)), \chi_i(x, \mathcal{A}) = \chi_i(T(x), \mathcal{A})$ *and* $\mathcal{A}(x)V_x^i = V_{T(x)}^i$ *for almost every* x*;*

3) $\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}^n(x)\| = \chi_i(x, \mathcal{A})$ *for every* $v \in V_x^i \setminus V_x^{i+1}$.

If the system (T, μ) *is ergodic then the functions* $x \mapsto k(x), \chi_i(x, \mathcal{A}), \dim V_x^i$ *are constant* μ*-almost everywhere.*

The conclusion of this theorem may be sharpened considerably when the map T is invertible, as long as we also assume that $\log^+ \|\mathcal{A}\|$ is integrable with respect to μ. Indeed, in this case instead of a flag one has a direct sum decomposition $\mathbb{R}^d = E_x^1 \oplus \ldots \oplus E_x^{k(x)}$ with

$$\mathcal{A}(x)E_x^i = E_{T(x)}^i, \quad \lim_{n\to\pm\infty} \frac{1}{n} \log \|\mathcal{A}^n(x)\| = \chi_i(x, \mathcal{A}) \text{ for every } v \in E_x^i \setminus \{0\}.$$

The flag and the decomposition are related through $V_x^i = E_x^i \oplus V_x^{i+1}$. This invertible version of the Oseledets theorem also asserts that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|det\mathcal{A}^n(x)\| = \Sigma_i \chi_i(x, \mathcal{A}) \dim E_x^i \ \text{ for } \mu - \text{almost every x.} \quad (2.4.1.1)$$

The identity in (2.4.1.1) is precisely the Lyapunov regularity condition for x.

The Oseledets theorem was first proven in [Ose]. Alternative arguments followed, by Raghunathan [Ragh], Ruelle [Ruell] and others. Dynamical systems proofs can be found in Walters [Walt] and Viana [V, Sections 4.2–4.3].

The numbers $\chi_i(x, \mathcal{A})$ and $\chi(\mu, \mathcal{A}) = \int \chi(., \mathcal{A}) d\mu$ are called the *Lyapunov exponents* and *Lyapunov exponents of measures* of the linear cocycle, respectively. The number $m_i = \dim V^i - \dim V^{i+1} (= \dim E^i$ in the invertible case) is called the *multiplicity* of the corresponding Lyapunov exponent $\chi_i(x, \mathcal{A})$. The *Lyapunov spectrum* is the set of Lyapunov exponents counted with multiplicity, that is, the ordered list $\chi_1(., \mathcal{A}) \geq \ldots \geq \chi_d(., \mathcal{A})$ where each exponent $\chi_i(., \mathcal{A})$ is repeated m_i times.

Kingman's subadditive theorem and the following lemma show the Lyapunov exponents of measures are upper semi continuous on $\mathcal{M}(X, T)$.

Lemma 2.4.2 ([M3, Appendix A])**.** *If* $g : X \to \mathbb{R} \cup \{\infty\}$ *is an upper semi-continuous, then the map from* $\mathcal{M}(X, T)$ *to* $\mathbb{R} \cup \{-\infty\}$ *given by* $\mu \mapsto \int g d\mu$ *is upper semi-continuous. If* (f_n) *is a subadditive sequence of upper semi-continuous functions from* X *to* $\mathbb{R} \cup \{-\infty\}$*, then the map from* $\mathcal{M}(X, T) \to \mathbb{R} \cup \{-\infty\}$ *given by* $\mu \mapsto \inf_{m\geq 1} \int f_m d\mu$ *is also upper semi-continuous.*

2.5 Thermodynamic formalism

In this section we will introduce the main tools in this thesis which comes from thermodynamic formalism.

Ergodic theory has its origins in statistical mechanics and the study of the long term behavior of systems of large numbers of particles. In such systems precise computation of the behavior of each particle may be unfeasible, but through the ergodic theorems one is able to gain an understanding of the long term of a typical point and link the macroscopic behavior of the system with the microscopic laws governing individual particles. When we refer here to a typical point, we mean almost every point with respect to some suitable measure invariant under the transformation, but this leads to the question, with respect to which measure should one use the ergodic theorem? The empirical data available to physicists led them to the conclusion that the Gibbs measure is the most suitable such measure. The subsequent body of work that followed connecting Gibbs measures with other analogues of notions from statistical mechanics such as pressure, equilibrium states and entropy all in one beautiful and interwoven theory is now called *thermodynamic formalism.*

The connections established by this theory have proved to be powerful tools in many areas of dynamical systems including its dimension theory, rates of mixing and statistical properties of dynamical systems. The monographs of Bowen [Bow] and Ruelle [Ru] provide classical expositions of thermodynamic formalism in the original settings in which it was developed.

2.5.1 Additive thermodynamic formalism

Let (X, T) be a topological dynamical system. A continuous function $\psi : X \to \mathbb{R}$ is a potential.

For any $n \in \mathbb{N}$, one can define a new metric d_n on X by

$$d_n(x, y) = \max\{d(T^k(x), T^k(y)) : k = 0, ..., n-1\}.$$

For any $\varepsilon > 0$ a set $E \subset X$ is said to be a (n, ε)-*separated subset* of X if $d_n(x, y) > \varepsilon$ for any two different points $x, y \in E$.

Using (n, ε)-separated subsets, we can define a thermodynamic object called the *pressure* $P(\psi)$ of ψ as follows:

$$P(\psi) = \lim_{\varepsilon \to o} \limsup_{n \to \infty} \frac{1}{n} \log \sup\{\sum_{x \in E} e^{S_n(\psi(x))} : E \text{ is } (n, \varepsilon)\text{-separated subset of X}\}.$$

When $\psi = 0$, the pressure $P(0)$ is topological entropy $h_{top}(T)$, which measures the complexity of the system (X, T).

The pressure satisfies the *variational principle:*

$$P(\psi) = \sup\{h_\mu(T) + \int f d\mu : \mu \in \mathcal{M}(X,T)\}.$$

Any invariant measure $\mu \in \mathcal{M}(X,T)$ achieving the supremum in the variational principle is called an *equilibrium state of* ψ. If the entropy map $\mu \mapsto h_\mu(T)$ is upper semi-continuous, then any potential has an equilibrium state. However, the existence, the finiteness, or the uniqueness of the equilibrium state for a given potential is a subtle question that depends on the system (X,T) as well as the potential ψ . On the other hand, there are specific settings where such questions have an affirmative answer. When (X,T) is a mixing hyperbolic system, and the potential ψ is a Hölder continuous, then the result of Bowen [B2] states that there exists a unique equilibrium state μ_ψ, which has the Gibbs property.

The topological pressure function $P : C(X) \to \mathbb{R}$ is convex and $q \mapsto P(q\psi)$ is uniformly Lipschitz continuous.

Assume that ψ is a Hölder continuous function. $A(t) := P(t\psi)$ is differentiable in t, and $A^{'}(t) := \int \psi d\mu_t$, where μ_t is the unique equilibrium state for $t\psi$ (which also has Gibbs properties). Using convexity properties of A one can argue that $A^{'}(t)$ takes all values in the interior of $\{\int \psi d\mu_t \mid \mu_t \text{ is invariant}\}$. In particular, there is t such that $A^{'}(t) = \alpha$ such that $h_{\mu_t}(T) = \sup\{h_\mu(T) \mid \int \psi d\mu = \alpha\}$. See for more information [PU, Chapter 3-5] and [F7, Chapter 11].

2.5.2 Subadditive thermodynamic formalism

The additive theory of thermodynamic formalism extends to the subadditive theory with suitable generalizations. A natural example of a subadditive potential is the singular value potential of a continuous $GL(k,\mathbb{R})$-cocycle $\mathcal{A}$ over Σ.

Let (X,T) be a topological dynamical system. We define the *subadditive pressure* of a subadditive potential $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ as

$$\begin{aligned} P(T,\Phi) &= \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} P_n(T,\Phi,\varepsilon) \\ &= \sup\{\sum_{x\in E} \phi_n(x) : E \text{ is a } (n,\varepsilon)\text{-separated subset of X}\} \end{aligned}$$

where the existence of the limit is guaranteed from the subadditivity of Φ. One also observe that $h_{top}(T) := P(T,1)$. For $t \in \mathbb{R}_+$, we denote $P_\Phi(t) = P(T,t\Phi)$.

Cao, Feng and Huang [CFH] extend the additive theory of the variational principle to the subadditive theory.

Theorem 2.5.1 ([CFH], Theorem 1.1)**.** *Let (X,T) be a topological dynamical system such that $h_{top}(T) < \infty$. Suppose that $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ is a subadditive potential on a compact metric space (X,T). Then for $t > 0$*

$$P_\Phi(t) = \sup\{h_\mu(T) + t\chi(\mu,\Phi)$$

$$: \mu \in \mathcal{M}(X,T), \chi(\mu,\Phi) \neq -\infty\}.$$

2.5.3 Almost additive thermodynamic formalism

In this subsection, we state a theorem that shows that we have the Bowen's result for almost additive sequences.

We say that a subadditive sequences $\Phi := \{\log \phi_n\}_{n=1}^{\infty}$ over (Σ,T) has *bounded distortion:* there exists $C \geq 1$ such that for any $n \in \mathbb{N}$ and $I \in \mathcal{L}(n)$, we have

$$C^{-1} \leq \frac{\phi_n(x)}{\phi_n(y)} \leq C$$

for any $x, y \in [I]$.

Lemma 2.5.2 ([P, Lemma 3.10])**.** *Let $\mathcal{A}$ be a Hölder continuous and fiber-bunched $GL(k,\mathbb{R})$-cocycle over (Σ,T). Then $\Phi_{\mathcal{A}}$ has bounded distortion.*

Theorem 2.5.3 ([Bar, Theorem 10.1.9])**.** *Let $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ be an almost additive sequence over a topologically mixing subshift of finite type (Σ,T). Assume that Φ has bounded distortion. Then:*

1. *There is a unique equilibrium measure for Φ,*
2. *there is a unique invariant Gibbs measure for Φ,*
3. *the two measures coincide and are ergodic.*

2.6 Fractal dimensions

Assume that (X,d) is a metric space. We define *covers* and *packings* of a set $F \subset X$ at some scale $\delta > 0$. A collection $\{U_i\}_{i\in I}$ of subsets of X will be called a δ-cover of F if each of the sets U_i is open and has diameter less than or equal to δ, and F is contained in the union $\bigcup U_{i\in I}$. Similarly, a collection $\{U_i\}_{i\in I}$ of subsets of X will be called a *centered δ-packing* of F if each of the sets $\{U_i\}_{i\in I}$ are disjoint closed balls with radius less than or equal to δ and centers in F. Analyzing the behavior of such covers and packings as δ converges to zero will be crucial in developing the theory of dimension.

Hausdorff dimension, named after Felix Hausdorff, who introduced the notion in 1918 [Hau], is intrinsically linked with *packing dimension*, named due to its use of packings rather than the covers used to define Hausdorff dimension, which was introduced many years later in 1982 by Claude Tricot [Tricot]. These two dimensions have probably received the most attention in the literature on fractals and have found their way into various different fields. They both have a convenient definition in terms of measures, which leads to a mathematically beautiful theory but can often make them very difficult to compute directly.

Let F be a subset of X. For $s > 0$, and $\delta > 0$ we define the $\delta-$ approximates-dimensional Hausdorff measure of F by

$$\mathcal{H}^s_\delta(F) = \inf\{\sum_{i\in I} |U_i|^s \ : \ \{U_i\}_{i\in I} \text{ is a countable } \delta - \text{cover of F}\}$$

and the $s-$dimensional Hausdorff (outer) measure of F by $\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F)$. The Hausdorff dimension of F is

$$\dim_H F = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s > 0 : \mathcal{H}^s(F) = \infty\}.$$

The reader surely observed how similar the definitions of Hausdorff dimension and topological entropy are. Indeed, they are both just special cases of a more general construction of Caratheodory, see [Carath].

If F is compact, then we may define the Hausdorff measure of F in terms of finite covers. The following is a list of basic properties which Hausdorff dimension satisfy:

Monotonicity: $\dim_H$ is said to be monotone if $E \subset F$, then $\dim_H E \leq \dim_H F$ for all $E, F \subset X$.

Finite stability: $\dim_H$ is said to be finitely stable if $\dim_H(E \cup F) = \max\{\dim_H E, \dim_H F\}$ for all $E, F \subset X$.

Countable stability: $\dim_H$ is said to be countably stable if $\dim_H \cup_i E_i = \sup_i \dim_H E_i$ for all countable collections of sets $\{E_i\}$ in X.

Stability under (bi-)Lipschitz maps: $\dim_H$ is said to be stable under Lipschitz maps if $dim_H f(E) \leq (=) \dim_H E$ for all $E \subset$X and all (bi-)Lipschitz maps f on X.

Open set property: $\dim_H$ is said to satisfy the open set property if for any bounded open set $U \subset \mathbb{R}^n, \dim_H U = n$.

Packing measure, defined in terms of packings, is a natural dual to Hausdorff measure, which was defined in terms of covers. For $s > 0$ and $\delta > 0$ we define the δ-approximates-dimensional packing pre-measure of F by

$$\mathcal{P}^s_\delta(F) = \sup\{\sum_{i\in I} |U_i|^s \ : \ \{U_i\}_{i\in I} \text{ s a countable centered } \delta - \text{packing of F}\}$$

and the s-dimensional packing pre-measure of F by $\mathcal{P}_0^s(F) = \lim_{\delta\to 0}\mathcal{P}_\delta^s(F)$. To ensure countable stability, the packing (outer) measure of F is defined by

$$\mathcal{P}^s(F) = \inf\{\sum_{i\in I}\mathcal{P}_0^s(F_i)\ :\ F\subset \cup_i F_i\}$$

and the packing dimension of F is

$$\dim_P F = \inf\{s\geq 0 : \mathcal{P}^s(F) = 0\} = \sup\{s > 0 : \mathcal{P}^s(F) = \infty\}.$$

Packing dimension satisfy Monotonicity, Finite stability, Countable stability, Stability under (bi-)Lipschitz maps and Open set property.

The lower and upper box dimensions of a set $F\subset X$ are defined by

$$\underline{\dim}_B(F) := \liminf_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta} \text{ and } \overline{\dim}_B(F) := \limsup_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta}$$

respectively, where $N_\delta(F)$ is the smallest number of sets required for a δ-cover of F. If $\underline{\dim}_B(F) = \overline{\dim}_B(F)$, then we call the common value the *box dimension* of F and denote it by $\dim_B F$. It is useful to note that we can replace N_δ with a myriad of different definitions all based on covering or packing the set at scale δ, see [F8, Section 3.1]. For example, $N_\delta(F)$ can be taken as the maximal size of a centered δ-packing of F. If defining box dimension in a non-compact space, then usually one restricts totally bounded sets in order to preclude the situation where $N_\delta(F) = \infty$.

Box dimension satisfy Monotonicity, Stability under (bi-)Lipschitz maps and Open set property.

One could try to redefine box dimension by breaking the setup into countably many bits, taking the supremum of the box dimension of the bits and then taking the infimum over the different ways of splitting the set up. Amazingly, this new definition simply returns the packing dimension. We obtain

$$\dim_P F = \inf\{\sup_i \dim_B F_i : F\subset \cup_{i\in I}F_i\}$$

where the infimum is taken over all countable partitions $\{F_i\}_{i\in I}$ of F, see [F8, Section 3.4]. This alternative definition for packing dimension has the following very useful consequence.

Lemma 2.6.1. *Let $F\subset X$ be a compact set such that for every open set $U\subset X$ which intersects F, we have $\dim_B(F\cap U) = \dim_B F$. Then, $\dim_P F = \overline{\dim}_B F$.*

Assume that (X, d_X) and (Y, d_Y) are two metric spaces. We have

$$\dim_H(X) + \dim_H(Y) \leq \dim_H(X\times Y) \leq \dim_H X + \dim_P Y \leq \dim_P(X\times Y).$$

See [Mat, Theorem 8.10].

For a Borel probability measure μ the Hausdorff dimension is defined as

$$\dim_H \mu = \inf\{\dim_H A : A \text{ Borel set such that } \mu(A) = 1\}.$$

The *upper and lower local dimensions* of a Borel probability measure μ at a point x in its support are defined by

$$\overline{\dim}_{\text{loc}}(\mu, x) := \limsup_{n\to\infty} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \underline{\dim}_{\text{loc}}(\mu, x) := \liminf_{n\to\infty} \frac{\log \mu(B(x,r))}{\log r}.$$

If the upper and lower local dimensions coincide, we call the common value the *local dimension* and denote it by $\dim_{\text{loc}}(\mu, x)$. This describes the rate at which the measure of a small ball about a μ-typical point scales as the radius of the ball is decreased. This notion is particularly important because if there exists a constant α such that the local dimension exists and equals α at μ almost all points then we say the measure μ is *exact dimensional* and in particular, if μ is exact dimensional then all the definitions of the dimension of a measure coincide with the exact dimension α.

One can also define the Hausdorff dimension of measure in the following way:

$$\dim_H \mu = \sup\{s : \underline{\dim}_{\text{loc}}(\mu, x) \geq s \textit{ for almost all } x\ \}.$$

See for more information [F7, Section 10.1].

If $\dim_{\text{loc}}(\mu, x) \geq \delta$ for a set of points of positive measure, then $\dim_H \mu \geq \delta$; this is known as Frostman's Lemma. Assume that $U := \{U_i\}_{i\in I}$ cover a set F. Let μ be a mass distribution [2] on F and suppose that for some s there are numbers $C > 0$ and $\delta > 0$ such that

$$\mu(U) \geq C|U|^s$$

for all sets U with $|U| \leq \delta$. Then $s \leq \dim_H(F) \leq \underline{\dim}_B(F) \leq \overline{\dim}_B(F)$. See [F8, Mass distribution principle 4.2].

We now present some results from measure theory which might be used in the thesis. If a measure μ is absolutely continuous with respect to a measure ν, we write $\mu \ll \nu$. The following proposition (e.g. [MMR, Lemma 2.4]) is useful to verify exact-dimensionality whenever we have a measure which is absolutely continuous with respect to an exact-dimensional measure

Lemma 2.6.2. *Assume ν is a non-null finite Borel measure on $\mathbb{R}^d$ with exact dimension α. Let μ be any non-null finite Borel measure μ on $\mathbb{R}^d$ with $\mu \ll \nu$. Then, μ is exact dimensional with exact dimension α.*

[2] A mass distribution on F is a measure with support contained in F such that $0 < \mu(F) < \infty$

2.7 Self affine IFS

We say that $f : \mathbb{R}^d \to \mathbb{R}^d$ is a contraction if there exists a contraction ratio $0 < C < 1$ such that for all $x, y \in \mathbb{R}^d$

$$|f(x) - f(y)| \leq C|x - y|.$$

One of the most important ways of constructing fractals is via *iterated function systems.* An iterated function system (IFS) is a finite collection $\{f_i\}_{i \in I}$ of contracting self-maps. It is a fundamental result in fractal geometry, dating back to Hutchinson's seminal 1981 paper [Hut], that for every IFS there exists a unique non-empty compact set Λ, called the *attractor,* which satisfies

$$\Lambda = \bigcup_{i \in I} f_i(\Lambda).$$

2.7.1 Symbolic coding of IFS and attractors

Typically, attractors of iterated function systems are studied by building a symbolic space from the index set $\mathcal{I}$, since the geometry of the symbolic space is more convenient to work with than the more complex geometry of the attractor. Let $\mathcal{F} = \{f_i\}_{i \in \mathcal{I}}$ be an iterated function system on compact metric space X. We will now briefly describe this technique and fix some notation which will be used throughout the thesis whenever a fixed IFS or a system with markov partition indexed by $\mathcal{I}$ is present. Let $\mathcal{I}^* = \bigcap_{k \geq 1} \mathcal{I}^k$ and denote the set of all finite sequences with entries in $\mathcal{I}$ and for

$$\mathbf{i} = (i_1, \ldots, i_k) \in \mathcal{I}^*$$

and write

$$f_{\mathbf{i}} = f_{i_1} \circ \cdots \circ f_{i_k}.$$

We denote by $\mathcal{I}^{\mathbb{N}}$ the set of all infinite $\mathcal{I}$-valued strings and for $\mathbf{i} \in \mathcal{I}^{\mathbb{N}}$ or $\mathcal{I}^l$ with $l > k$ write $\mathbf{i}|k \in \mathcal{I}^k$ to denote the restriction of $\mathbf{i}$ to its first k entries.

Then, we define a natural projection

$$\rho : \mathcal{I}^{\mathbb{N}} \to \Lambda$$

from the symbolic space to the geometric space by

$$\rho(\mathbf{i}) = \bigcap_{k \in \mathbb{N}} f_{\mathbf{i}|k}(X).$$

Self-affine sets are attractors of IFS's where all of the maps are contracting affine self-maps on some Euclidean space. An *affine map* is the map $T : X \to Y$,

X and Y vector spaces, of the form $T(x) = Ax + v$, where $A \in \mathrm{Lin}(X, Y)$ and $v \in Y$. Self-affine sets are notoriously difficult to handle in comparison with self-similar, meaning that IFS consists solely of similarity transformation sets, and there are still many fascinating open problems in the area.

The *singular values* of a linear map, $A : \mathbb{R}^n \to \mathbb{R}^n$, are the positive square roots of the eigenvalues of $A^T A$. Viewed geometrically, these numbers are the lengths of the semi-axes of the image of the unit ball under A. Thus, roughly speaking, the singular values correspond to how much the map contracts (or expands) in different directions. For $s \in [0, n]$ define the singular value function $\varphi^s(A)$ by

$$\varphi^s(A) = \sigma_1\sigma_2 \ldots \sigma_{\lceil s\rceil-1}\sigma_{\lceil s\rceil}^{s-\lceil s\rceil+1} \tag{2.7.1.1}$$

where $\sigma_1 > \cdots > \sigma_n$ are the singular values of A. This function has played a important role in the study of self-affine sets over the past 25 years. Let $\{A_i : i \in \mathcal{I}\}$ be a finite collection of contracting linear self-maps on $\mathbb{R}^n$, write $m = |\mathcal{I}|$ and let

$$d = d(A_i : i \in A_i) = \inf\{s : \sum_{k=1}^{\infty} \sum_{\mathcal{I}^k} \varphi^s(A_{i_1} \ldots A_{i_k}) < \infty\}. \tag{2.7.1.2}$$

This number is called the *affinity dimension* of F and is always an upper bound for the upper box dimension of F, see [Fal]. Moreover, by considering a natural cover of F. Falconer proved that for ‘typical’ translations it was equal to the Hausdorff dimension of the set.

Theorem 2.7.1. *Let $\{A_i\}_{i=1}^k$ be a collection of $n \times n$ matrices where each A_i satisfies the bound on its matrix norm $\sigma_1(A_i) < \frac{1}{2}$. Then for Lebesgue almost all translations $(t_1, \ldots, t_k) \in \mathbb{R}^{kn}$, the attractor F of the self-affine IFS $F = \{A_i + t_i : i = 1, \ldots, k\}$ satisfies*

$$\dim_H F = \min\{n, d\}.$$

In fact, initially the above result was proved in [Fal] with the stronger assumption that all the norms $\sigma_1(A_i) < \frac{1}{3}$, but in [So], Solomyak weakened the condition to the current form. Moreover, an upper bound of $1/2$ was proved to be sharp by an example of Przytycki and Urbański in [PU1].

2.8 Smooth dynamics

We recall some basic definitions, properties of Anosov diffeomorphisms and partially hyperbolic dynamics which are going to be useful throughout this thesis. During this section we use [KH, Section 6.4] and [CP].

Definition 2.8.1. Let X be a connected smooth manifold. A diffeomorphism $f : X \to X$ is called an *Anosov diffeomorphism or uniformly hyperbolic* if there is an invariant decomposition of the tangent bundle TX as a direct sum of continuous Df-invariant sub bundles E^s_x and E^u_x such that, for some appropriate Riemannian metric,

$$\|Df^n_x(v^s)\| \leq C\lambda^n\|v^s\| \text{ and } \|Df^{-n}_x(v^u)\| \leq C\lambda^n\|v^u\|,$$

for all $x \in X$ and for any pair of unit vectors $v^s \in E^s_x, v^u \in E^u_x$, where $0 < \lambda < 1$ and $C > 0$ are both constants.

For example, a diffeomorphism $T : \mathbb{T}^2 \to \mathbb{T}^2$ induced by the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ is an Anosov diffeomorphism.

We proceed to show a basic proposition about the dependence of E^s_x and E^u_x on x.

Proposition 2.8.2. *Let $f : X \to X$ be an Anosov diffeomorphism. Then, the subspaces E^s_x and E^u_x depend continuously on x.*

We state fundamental result about stable and unstable manifolds for an Anosov diffeomorphism. Let d be the Riemannian distance function.

Theorem 2.8.3 (Stable Manifold Theorem)**.** *Let $f : X \to X$ be an Anosov diffeomorphism of class C^k. Then there exist $\varepsilon > 0$ and $0 < \lambda < 1$ such that for each $0 < \varepsilon < \varepsilon_0 \in X$, the local stable manifold*

$$W^s_{\text{loc}}(x) = \{y \in X : d(f^n(x), f^n(y)) \leq \varepsilon \, for \; all \, n \geq 0\},$$

and the local unstable manifold

$$W^u_{\text{loc}}(x) = \{y \in X : d(f^{-n}(x), f^{-n}(y)) \leq \varepsilon \, for \; all \, n \geq 0\},$$

are C^k embedded disks tangent at x to E^s_x and E^u_x respectively. In addition,

- $f(W^s_{\text{loc}}(x)) \subset W^s_{\text{loc}}(f(x))$ *and* $f^{-1}(W^u_{\text{loc}}(x)) \subset W^u_{\text{loc}}(f^{-1}(x))$;
- $d(f(x), f(y)) \leq \lambda d(x, y)$ *for all* $y \in W^s_{loc}(x)$;
- $d(f^{-1}(x), f^{-1}(y)) \leq \lambda d(x, y)$ *for all* $y \in W^u_{loc}(x)$;
- $W^s_{\text{loc}}(x)$ *and* $W^u_{\text{loc}}(x)$ *vary continuously with the point x in the C^k topology.*

Furthermore, the global stable and unstable manifolds of x,

$$W^s(x) = \cup_{n=0}^{\infty} f^{-n}(W^s_{\text{loc}}(f^n(x)) \text{ and } W^u(x) = \cup_{n=0}^{\infty} f^n(W^s_{\text{loc}}(f^{-n}(x)),$$

are smoothly immersed submanifolds of X.

A well-known property of Anosov dynamics is their *local product structure*. More precisely, there is a constant $\delta_1 > 0$ such that for every $x, y \in X$ which satisfy $d(x, y) < \delta_1$ the intersection $W^u_{\text{loc}}(x) \cap W^s_{\text{loc}}(y)$ consists of a unique point denoted by $[x, y]$. In fact, for ε small enough the local stable manifold $W^s_\varepsilon(x)$ and the local unstable manifold $W^s_\varepsilon(x)$ have transversal intersection at x and these manifolds vary C^1-continuously respect to x. As a result, we get the local product structure.

2.8.1 Closing property

Another well-known property of Anosov dynamics is *closing property.*

A sequence $x_1, ..., x_n = x_0$ of points is called a *periodic ε-pseudo-orbit* if $d(f(x_k), x_{k+1}) < \varepsilon$ for all $k = 1, ..., n$. A homeomorphism $f : X \to X$ satisfies the closing property if there exist two positive constants C, δ_0 such that for $\varepsilon < \delta_0$ any periodic $\varepsilon-$pseudo-orbit $x_0, x_1, ..., x_n = x_0$, there is a periodic point p such that $f^n(p) = p$ and $d(f^k p, x_k) < C\varepsilon$, for every $k \in \{0, 1, ..., n\}$.

If a homeomorphism f satisfies the closing property and $x \in X$ satisfied $(x, f^n x) < \delta_0$, then there is a periodic point $p = f^n(p)$ such that $d(f^k x, f^k p) < C\varepsilon$ for every $k \in \{0, 1, ..., n\}$.

Theorem 2.8.4 (Anosov closing lemma)**.** *Every Anosov diffeomorphism $f : X \to X$ satisfies the closing property.*

Note that shifts of finite type, Axiom A diffeomorphism, and hyperbolic homeomorphism are particular systems satisfying the Anosov closing property.

2.8.2 Partially hyperbolic dynamics

Partial hyperbolicity is a relaxed form of uniform hyperbolicity which intends to address larger families of dynamics. A main goal of their study consists in understanding how the properties of uniformly hyperbolic systems extends.

We consider M a closed connected d-dimensional Riemannian manifold and let TM its tangent bundle. We also consider f in the space $\text{Diff}^r M)$ of C^r-diffeomorphisms endowed with the C^r-topology, $r \geq 1$.

Definition 2.8.5. A partially hyperbolic set for f is a compact f-invariant set K whose tangent bundle admits a splitting into three continuous vector subbundles $T_K M = E^u \bigoplus E^c \bigoplus E^s$ which satisfy:

- the splitting is dominated,
- E^s is uniformly contracted, E^u is uniformly expanded, one of them is non-trivial.

A splitting $T_K M = E_1 \bigoplus \ldots \bigoplus E_k$ is a *dominated splitting* if and only if:

- **Invariance:** The bundles are Df -invariant. This means that for every $x \in K$ and $1 \leq i \leq k$ one has $D_x f(E_i(x)) = E_i(f(x))$.

- **Domination:** There exists constants $C > 0$ and $\lambda \in (0,1)$ such that, for every $1 \leq i \leq k-1$, for every $x \in K$ and vectors $u \in E_i(x) \setminus \{0\}$ and $u \in E_{i+1}(x) \setminus \{0\}$ one has:

$$\frac{\|D_x f^n u\|}{\|u\|} \leq C\lambda^n \frac{\|D_x f^n v\|}{\|v\|}, \quad \forall n \geq 0. \tag{2.8.2.1}$$

Domination can be also expressed by saying that for any $x \in K$ and $1 \leq i \leq k-1$ one has that $\|D_{|E_i(x)} f^n u\| \leq C\lambda^n \|(D_{|E_{i+1}(x)} f^n)^{-1}\|^{-1}$.

Remark 2.8.6. *a) If $k = 1$ we say that the splitting is trivial. Sometimes, when one says that an f -invariant subset admits a dominated splitting one implicitly assumes that it is not trivial.*

b) One can replace condition (2.8.2.1) by asking for the existence of $n > 0$ such that for any $x \in K$ and vectors $u \in E_i(x) \setminus \{0\}$ and $v \in E_{i+1}(x) \setminus \{0\}$ one has:

$$\frac{\|D_x f^n u\|}{\|u\|} \leq \frac{1}{2} \frac{\|D_x f^n v\|}{\|v\|}.$$

In any case, in such a situation one says that E_{i+1} dominates E_i and one someone denotes this as $E_1 \oplus_< \ldots \oplus_< E_k$ to emphasize the order of the domination.

- *If one replaces the bundles E_i, E_{i+1} by their direct sum $E_i \oplus E_{i+1}$ the splitting remains dominated.*

2.8.3 Convex cone

We adopt the convention that if V is a vector space, a convex cone C in V is a subset such that there exists non-degenerate quadratic form Q_C such that

$$C = \{v \in V : Q_C(v) \geq 0\}.$$

The interior of a convex cone is $\operatorname{interior} C = \{v \in V : Q_C(v) > 0\} \cup \{0\}$.

A cone-field C on $K \subset M$ is then a choice of a convex cone $C_x \subset T_x M$ for each point in M such that in local charts the quadratic forms can be chosen in a continuous way and have the same signature (d_-, d_+).

Equivalently, a cone-field C in K is given by:

- a (not necessarily invariant) splitting $T_K M = \hat{E} \oplus \hat{F}$ into continuous subbundles whose fibers have dimension d_- and d_+ respectively,
- a continuous family of Riemannian norms $\|\cdot\|$ defined on $T_K M$ (not necessarily the ones given by the underlying Riemannian metric).

In this setting, for $x \in K$, one associates

- the convex cone $C_x = \{v = v_{\hat{E}} + v_{\hat{F}} \in T_x M, \|v_{\hat{F}}\| \geq \|v_{\hat{E}}\|\}$,
- the dual convex cone $C_x^* = \{v = v_{\hat{E}} + v_{\hat{F}} \in T_x M, \|v_{\hat{E}}\| \geq \|v_{\hat{F}}\|\}$.

The dimension $\dim C$ of the cone-field C is the dimension d_+ of the bundle $\hat{F}$.

We say that a cone-field C defined in K is Df-contracted if there exists $N > 0$ such that for every $x \in K \cap \ldots \cap f^{-N}(K)$ one has that

$$D_x f^N(C_x) \subset \text{interior}(C_{f^N(x)})$$

(Equivalently, the dual cone field C_x^* is Df^{-1}-contracted).

Theorem 2.8.7 ([CP, Theorem 2.6]). *Assume that $f \in \mathrm{Diff}^r(M)$. Let K be an invariant compact set and fix $d_+ \geq 1$. Then K is endowed with a Df-contracted cone-field C with dimension d_+ if and only if there exists a dominated splitting $T_K M = E \oplus_< F$ with $d_+ =$ dim(F).*

2.9 Hilbert metric

Let V be a vector space over the reals.

Definition 2.9.1. Fix a convex cone $C \subset V$. Given $v, w \in C$, let

$$\alpha(v,w) = \sup\{\lambda > 0 | w - \lambda v \in C\}, \quad \beta(v,w) = \inf\{\mu > 0 | \mu v - w \in C\}, \tag{2.9.1}$$

with $\alpha = 0$ and/or $\beta = \infty$ if the corresponding set is empty. The cone distance between v and w is

$$d_c(v,w) = \log \frac{\beta(v,w)}{\alpha(v,w)}. \tag{2.9.2}$$

The distance d_c is called Hilbert projective (pseudo) metric.

Several remarks are now in order. First we observe that although V may be infinite-dimensional, the distance $d_c(v,w)$ is completely determined in terms of the two-dimensional subspace spanned by v and w, and in particular by the points shown in Figure 2, the lines $0A$ and $0B$ are the boundary of this two-dimensional cross-section of C. The lines $0X$ and wY are parallel, as are the lines $0A$ and wX; then we have

$$\alpha = \frac{|wY|}{|0v|}, \text{ and } \beta = \frac{|0X|}{|0v|}.$$

An alternate description of d_C is available in terms of this more geometric description. Let J be the line through v and w and let A, B be the points where this line intersects the boundary of C. We see from Figure 2 that the triangles BYw and $B0v$ are similar, so

$$\alpha = \frac{|wY|}{|0v|} = \frac{|Bw|}{|Bv|}.$$

Furthermore, $v0A$ and vXw, are similar so

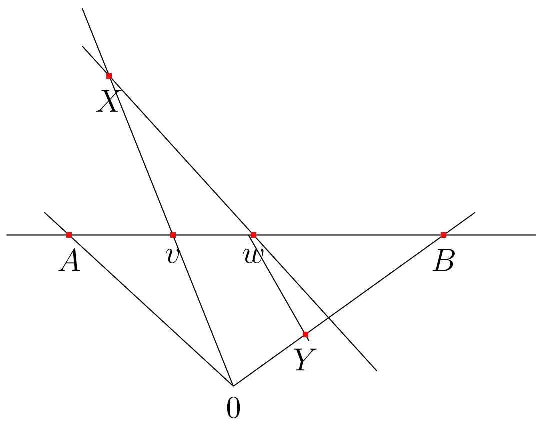


Figure 2: Determining the cone distance between v and w

$$\beta = \frac{|0X|}{|0v|} = 1 + \frac{|vX|}{|0v|} = 1 + \frac{|wv|}{|Av|} = \frac{|Aw|}{|Av|}.$$

Thus d_C can be given in terms of the *cross-ratio* of the points v, w, A, B :

$$\frac{\beta}{\alpha} = \frac{|Aw|}{|Av|}\frac{|Bv|}{|Bw|} := (v, w; A, B).$$

We have

$$d_C(v, w) = \log(v, w; A, B).$$

Note that it is possible that the line J does not intersect the boundary of C twice; this corresponds to the the case when either $\alpha = 0$ or $\beta = \infty$ (or both) in and in this case $d_C(v, w) = \infty$.

Moreover, when $\alpha = \beta$ (that is, when $v = cw$, they are colinear), the Hilbert metric then gives 0. Because of this phenomenon, d_C is not a true metric (it is a pseudometric metric). However, d_C is a projective metric on $C/\sim$ (the equivalence classes of C for the relation $x \sim y$ if $x = \lambda y$ where $\lambda \in \mathbb{R}_+$).

An important property of the Hilbert metric is the following theorem, due to Birkhoff, which states that a linear map from one convex cone to another is a contraction whenever its image has finite diameter (see for more information [1]).

Theorem 2.9.2. *Let $C_1 \subset V_1$ and $C_2 \subset V_2$ be convex cones, and $L : V_1 \to V_2$ be a linear map such that $L(C_1) \subset C_2^o$. Assume that $\triangle := \sup_{\hat{v},\hat{\psi} \in L(C_1} d_{C_2}(\hat{v}, \hat{w})$. Then for all $v, w \in C_1$, we have*

$$d_{C_2}(Lv, Lw) \leq \tanh(\frac{\triangle}{4}) d_{C_1}(v, w),$$

where we use the convention that $\tanh \infty = 1$.

We are also going to use the following lemma.

Lemma 2.9.3. *Let V be a finite dimensional vector space. Suppose that C_1 and C_2 are two convex cones in V such that $C_1 \subset C_2^o$ and d_{C_2} is the Hilbert metric on C_2. Then C_1 is bounded in metric d_{C_2}.*

Proof. Let us denote d as the usual distance on the projective space. Since $C_1 \subset C_2^o$, $d(C_1, \partial C_2) > 0$. Hence, for every $v, w \in C_1$ the distances $d(A, v), d(B, w)$ are uniformly bounded from below by $c_1 = d(C_1, \partial C_2)$, where A, B are the intersections of the line $\overline{vw}$ with ∂C_2 (see Figure 2). On the other hand, $d(v, w)$ is uniformly bounded from above by $c_2 = \mathrm{diam}_d(C_1)$. Thus, $d_{C_2}(v, w) \leq \log((c_1 + c_2)/c_1)$. $\square$

Chapter 3

Multifractal formalism

We recall that we are interested in the linear cocycles generated by a topologically mixing subshift of finite type $T : \Sigma \to \Sigma$ and a Hölder continuous function $\mathcal{A} : \Sigma \to GL(k, \mathbb{R})$.

3.1 Multilinear algebra

We recall some basic facts about the exterior algebra. We use them for studying the singular value function.

We denote by $\sigma_1, ..., \sigma_k$ the singular values of the matrix $\mathcal{A}$, which are the square roots of the eigenvalues of the positive semi definite matrix $\mathcal{A}^*\mathcal{A}$ listed in decreasing order according to multiplicity.

Assume that $\{e_1, .., e_k\}$ is the standard orthogonal basis of $\mathbb{R}^k$ and define

$$\wedge^l \mathbb{R}^k := \{e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_l} : 1 \leq e_{i_1} \leq e_{i_2} \leq ... \leq e_{i_l} \leq k\}$$

for all $l \in \{1, ..., k\}$ with the convention that $\wedge^0 \mathbb{R}^k = \mathbb{R}$. It is called the *l-th exterior power of* $\mathbb{R}^k$.

We are interested in the invertible matrices $GL(k, \mathbb{R})$. We consider induced topology $\mathbb{R}^{k^2}$ for it. For $A \in GL(k, \mathbb{R})$, we define an invertible linear map $A^{\wedge l} : \wedge^l \mathbb{R}^k \to \wedge^l \mathbb{R}^k$ as follows

$$(A^{\wedge l}(e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_l})) = Ae_{i_1} \wedge Ae_{i_2} \wedge ... \wedge Ae_{i_l}.$$

$\wedge^l \mathbb{R}^k$ is represented by a $\binom{k}{l} \times \binom{k}{l}$ whose entries are the $l \times l$ minors of A. It can also show that

$$(AB)^{\wedge l} = A^{\wedge l} B^{\wedge l}.$$

The singular values of $A^{\wedge l}$ are the product $\sigma_{i_1}(A)...\sigma_{i_l}(A)$. In addition,

$$\|A^{\wedge l}\| = \sigma_1(A)...\sigma_l(A).$$

3.2 The maximal and minimal Lyapunov exponent

We recall for $\mathcal{A} : \Sigma \to GL(k, \mathbb{R})$ and $I \in \mathcal{L}$

$$\|\mathcal{A}(I)\| := \max_{x\in[I]} \|\mathcal{A}^{|I|}(x)\|.$$

We define a positive continuous function $\{\varphi_{\mathcal{A},n}\}_{n\in\mathbb{N}}$ on Σ such that

$$\varphi_{\mathcal{A},n}(x) = \|\mathcal{A}^n(x)\|.$$

We denote by $\Phi_{\mathcal{A}}$ the subbadditive potential $\{\log \varphi_{\mathcal{A},n}\}_{n=1}^{\infty}$.

We recall the definition of the maximal Lyapunov exponent of linear cocycles

$$\beta(\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} \|\mathcal{A}^n(x)\|.$$

Morris [M2] showed that the maximal Lyapunov exponent is equal to the supremum of the Lyapunov exponents of measure over invariant measures. That means,

$$\beta(\mathcal{A}) = \sup_{\mu\in\mathcal{M}(X,T)} \chi(\mu, \mathcal{A}). \tag{3.2.1}$$

Feng and Huang [FH] gave a different proof of it.

Let us recall the set of maximizing measures of $\mathcal{A}$ to be the set of measures on X given by

$$\mathcal{M}_{\max}(\mathcal{A}) := \{\mu \in \mathcal{M}(X, T),\ \ \beta(\mathcal{A}) = \chi(\mu, \mathcal{A})\}.$$

We also recall the definition of the minimal Lyapunov exponents of linear cocycles as follows

$$\alpha(\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \inf_{x\in X} \|\mathcal{A}^n(x)\|.$$

Similarly, the set of minimizing measures is defined as follows

$$\mathcal{M}_{\min}(\mathcal{A}) := \{\mu \in \mathcal{M}(X, T),\ \ \alpha(\mathcal{A}) = \chi(\mu, \mathcal{A})\}.$$

We remark that supremum (3.2.1) is attained, so $\mathcal{M}_{\max}$ is non-empty set. But, $\mathcal{M}_{\min}$ is not necessarily non-empty.

Similarly, one can define the above definitions for subadditive potentials.

We define sum of top l Lyapunov exponents as follows

$$\chi_l(\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \varphi_{\mathcal{A}^{\wedge l},n}(x),$$

if the limit exists.

Similarly, we define sum of top l Lyapunov exponents of measure as follows

$$\chi_l(\mu, \mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \int \log \varphi_{\mathcal{A}^{\wedge l},n}(x) d\mu(x),$$

for $\mu \in \mathcal{M}(X, T)$.

We are mainly concerned with the distribution of the Lyapunov exponents of $\mathcal{A}$. More precisely, for any $\alpha \in \mathbb{R}$, define

$$E_{\mathcal{A}}(\alpha) = \{x \in X, \chi_1(\mathcal{A}) = \alpha\},$$

which is called the α-level set of $\chi_1(\mathcal{A})$.

We also define the higher dimensional of level set of all of Lyapunov exponents as follows

$$E_{\mathcal{A}}(\vec{\alpha}) = \{x \in X, \chi_l(\mathcal{A}) = \alpha_l\},$$

for $1 \leq l \leq k$.

We denote $E(\alpha) = E_{\mathcal{A}}(\alpha)$, when there is no confusion about $\mathcal{A}$.

We denote

$$\vec{\Phi}_{\mathcal{A}} := (\{\log \varphi_{\mathcal{A},n}\}_{n=1}^{\infty}, \{\log \varphi_{\mathcal{A}^{\wedge 2},n}\}_{n=1}^{\infty}, ..., \{\log \varphi_{\mathcal{A}^{\wedge k},n}\}_{n=1}^{\infty}).$$

We say that $\vec{\Phi}_{\mathcal{A}}$ is *(simultaneously) quasi-multiplicative* if there exist $C > 0$ and $m \in \mathbb{N}$ such that for every $I, J \in \mathcal{L}$, there exists $K \in \mathcal{L}$ with $|K| \leq m$ such that $IKJ \in \mathcal{L}$ and

$$\|\mathcal{A}^{\wedge i}(IKJ)\| \geq C\|\mathcal{A}^{\wedge i}(I)\|\|\mathcal{A}^{\wedge i}(J)\|,$$

for $1 \leq i \leq k$.

3.3 Thermodynamic Formalism

3.3.1 Legendre transform

Assume that $f : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$ is a convex function that is not identically equal to $-\infty$. The Legendre transform of f is the function f^* of a new variable t, defined by

$$t \mapsto -f^*(-t) := \inf\{f(x) - tx : x \in \mathbb{R}^k\},$$

where right hand side is scalar product.

It is easy to show that f^* is a convex function and not identically equal to $-\infty$. Let f^{**} be the Legendre transform of f^*. The following result is well known (cf. [Roc, Theorem 12.2]).

Theorem 3.3.1. *Assume that $f : \mathbb{R}^k \to \mathbb{R} \cup \{\infty\}$ is convex and not identically equal to $-\infty$. Let $x \in \mathbb{R}^k$. Suppose that f is lower semi continuous at x, i.e., $\liminf_{y\to x} f(y) \geq f(x)$. Then $f^{**}(x) = f(x)$.*

Feng and Huang [FH, Corollary 2.5] proved the following corollary as an application the above theorem.

Corollary 3.3.2. *Assume that A is a non-empty, closed and convex set in $\mathbb{R}^k$ and let $g : A \to \mathbb{R}$ be a concave function. Set*

$$W(x) = \sup\{g(a) + ax : a \in A\}, \quad x \in \mathbb{R}^k$$

and

$$G(a) = \inf\{W(x) - ax : x \in \mathbb{R}^k\}, \quad a \in A.$$

Finally, if g is upper semi continuous at $a \in A$, then $G(a) = g(a)$.

3.3.2 Thermodynamic Formalism for subadditive potentials

Assume that (X, T) is a topological dynamical systems.

Let $\vec{q} = (q_1, ..., q_k) \in \mathbb{R}^k_+$, and $\vec{\Phi} = (\Phi_1, ..., \Phi_k) = (\{\log \phi_{n,1}\}_{n=1}^{\infty}, ..., \{\log \phi_{n,k}\}_{n=1}^{\infty})$. Assume that $\vec{q}.\vec{\Phi} = \sum_{i=1}^{k} q_i \Phi_i$ is a subadditive potential $\{q_i \log \phi_{n,i}\}_{n=1}^{\infty}$. We can write topological pressure, maximal Lyapunov exponent, and minimal Lyapunov exponent of $\vec{\Phi}$, respectively

$$P_{\vec{\Phi}}(\vec{q}) = P(T, \vec{q}.\vec{\Phi}), \quad \vec{\beta}(\vec{\Phi}) = \beta(\sum_{i=1}^{k} \Phi_i), \quad \vec{\alpha}(\vec{\Phi}) = \alpha(\sum_{i=1}^{k} \Phi_i).$$

For $\mu \in \mathcal{M}(X, T)$, we write

$$\chi(\mu, \vec{\Phi}) = (\chi(\mu, \Phi_1), ..., \chi(\mu, \Phi_k)),$$

where $\chi(\mu, \Phi_i) = \lim_{n\to\infty} \frac{1}{n} \int \log \phi_{n,i}(x) d\mu(x)$ for $i = 1, ..., k$.

Theorem 3.3.3 ([FH, Theorem 1.2])**.** *Let (X, T) be a topological dynamical systems such that $h_{top}(T) < \infty$. Assume that $\vec{\Phi}$ is a subadditive potential on the compact metric space X. Then the pressure function $P_{\vec{\Phi}}(\vec{t})$ is a continuous real convex function on $(0, \infty)$. Furthermore, $P'_{\vec{\Phi}}(\infty) := \lim_{t\to\infty} \frac{P_{\vec{\Phi}}(\vec{t})}{\vec{t}} = \vec{\beta}(\vec{\Phi})$.*

We recall the definition of topological pressure by the following variational principle.

Theorem 3.3.4 ([CFH, Theorem 1.1]). *Let (X,T) be a topological dynamical systems such that $h_{top}(T) < \infty$. For $\vec{t} \in \mathbb{R}_+^k$, suppose that $\vec{\Phi}$ is a subadditive potential on the compact metric space X. Then*

$$P_{\vec{\Phi}}(\vec{t}) = \sup\{h_\mu(T) + \vec{t}.\chi(\mu, \vec{\Phi})$$

$$: \mu \in \mathcal{M}(X,T), \chi(\mu, \vec{\Phi}) \neq -\infty\}.$$

Let $\vec{t} \in \mathbb{R}_+^k$, we denote by $\mathrm{Eq}(\vec{\Phi}, \vec{t})$ the collection of invariant measures μ such that

$$h_\mu(T) + \vec{t}.\chi(\mu, \vec{\Phi}) = P_{\vec{\Phi}}(\vec{t}).$$

If $\mathrm{Eq}(\vec{\Phi}, \vec{t}) \neq \emptyset$, then each element $\mathrm{Eq}(\vec{\Phi}, \vec{t})$ is called an *equilibrium state* for $\vec{t}.\vec{\Phi}$.

In the remaining part of this section, we recall some theorems about multifractal formalism for subadditive potentials.

Theorem 3.3.5 ([FH, Proposition 3.2]). *Assume that $h_{top}(T) < \infty$. Then, $P_{\vec{\Phi}}(.)$ is a real continuous convex function on $\mathbb{R}_+^k$ and*

$$\partial P(\mathbb{R}_+^k) \subset (-\infty, \beta(\Phi_1)] \times ... \times (-\infty, \beta(\Phi_k)].$$

Theorem 3.3.6 ([FH, Theorem 1.1]). *Let (X,T) be a topological dynamical system such that the topological entropy $h_{top}(T)$ is finite. Then $E(\beta(\Phi)) \neq \emptyset$. Moreover,*

$$\begin{aligned} h_{top}(T, E(\beta(\Phi))) &= \sup\{h_\mu(T) : \mu \in \mathcal{M}(X,T), \chi(\mu, \Phi) = \beta(\Phi)\} \\ &= \sup\{h_\mu(T) : \mu \in \mathcal{E}(X,T), \chi(\mu, \Phi) = \beta(\Phi)\}. \end{aligned}$$

The topological pressure is related to Lyapunov exponents in the following way.

Proposition 3.3.7 ([FH, Theorem 3.3]). *Let (X,T) be a topological dynamical system such that the entropy map $\mu \mapsto h_\mu(T)$ is upper semi-continuous and $h_{top}(T) < \infty$. For $t \in \mathbb{R}_+^k$, suppose that $\vec{t}.\vec{\Phi}$ is a subadditive potential on the compact metric space X. Then,*

$$\partial P_{\vec{\Phi}}(\vec{t}) = \{\chi(\mu_{\vec{t}}, \vec{\Phi}) : \mu \in \mathrm{Eq}(\vec{\Phi}, \vec{t})\}. \tag{3.3.2.1}$$

Moreover, $\mathrm{Eq}(\vec{\Phi}, \vec{t})$ is a non-empty compact convex subset of $\mathcal{M}(X,T)$, for any $t \in \mathbb{R}_+^k$. Furthermore, the above results hold for $t \in \mathbb{R}^k$ when $\vec{\Phi}$ is an almost additive.

The following lemma shows that we can always approximate the Lyapunov exponent of the equilibrium measure by the Lyapunov exponent of the ergodic measure.

Lemma 3.3.8 ([FH, Lemma 4.7]). *Suppose that $h_{top}(T) < \infty$, and $\vec{t} \in \mathbb{R}^k_+$. Let $\vec{\alpha} \in \partial P^e_{\vec{\Phi}}(\vec{t})$. Then for any $\varepsilon > 0$, there is a $\nu \in \mathcal{E}(X,T)$ such that*

$$|\chi(\nu, \vec{\Phi}) - \vec{\alpha}| < \varepsilon, \text{ and} |h_\nu(T) - (P_{\vec{\Phi}}(\vec{t}) - \vec{\alpha}.\vec{t})| < \varepsilon.$$

Theorem 3.3.9 ([FH, Theorem 4.8]). *Keep the assumption of Theorem (3.3.6), we also assume that the entropy map $\mu \mapsto h_\mu(T)$ is upper semi-continuous on $\mathcal{M}(X,T)$. If $t \in \mathbb{R}^k_+$ such that $\vec{t}.\vec{\Phi}$ has a unique equilibrium state $\mu_{\vec{t}} \in \mathcal{M}(X,T)$, then $\mu_{\vec{t}}$ is ergodic, $\nabla P_{\vec{\Phi}}(\vec{t}) = \chi(\mu_{\vec{t}}, \vec{\Phi})$, $E(\nabla P_{\vec{\Phi}}(\vec{t})) \neq \emptyset$ and $h_{top}(T, E(\nabla P_{\vec{\Phi}}(\vec{t}))) = h_{\mu_{\vec{t}}}(T)$.*

We denote by $\mathcal{M}(X)$ the space of all Borel probability measure on X with weak* topology.

Theorem 3.3.10 ([CFH, Lemma 2.3]). *Suppose $\{\nu_n\}_{n=1}^\infty$ is a sequence in $\mathcal{M}(X)$ and $\Phi = \{\log \phi_n\}_{n=1}^\infty$ is a subadditive potential on the compact metric space X. We form the new sequence $\{\mu_n\}_{n=1}^\infty$ by $\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} \nu_n o T^i$. Assume that μ_{n_i} converges to μ in $\mathcal{M}(X)$ for some subsequence $\{n_i\}$ of natural numbers. Then $\mu \in \mathcal{M}(X,T)$ and*

$$\limsup_{i\to\infty} \frac{1}{n_i} \int \log \phi_{n_i}(x) d\nu_i(x) \leq \chi(\mu, \Phi). \tag{3.3.2.2}$$

Given an almost additive potential $\Phi = \{\log \phi_n\}_{n=1}^\infty$. Feng and Huang [FH, Lemma A.4] proved the following lemma:

Lemma 3.3.11. *Let $\mu \in \mathcal{M}(X,T)$. Then, the map $\mu \mapsto \chi(\mu, \Phi)$ is continuous on $\mathcal{M}(X,T)$.*

3.4 Generic cocycles

3.4.1 Fiber bunched cocycles

We recall that $T : \Sigma \to \Sigma$ is a topologically mixing subshift of finite type. We say that $\mathcal{A} : \Sigma \to GL(k, \mathbb{R})$ is a r-Hölder continuous function, if there exists $C > 0$ such that

$$\|\mathcal{A}(x) - \mathcal{A}(y)\| \leq C d(x,y)^r \ \ \forall x, y \in \Sigma. \tag{3.4.1.1}$$

We denote by $H^r(\Sigma, GL(k, \mathbb{R}))$ the set of r-Hölder continuous functions. We also show by $H^r(\Sigma)$, when there is no confusion about $GL(k, \mathbb{R})$.

We denote by $h_r(\mathcal{A})$ the smallest constant C in (3.4.1.1). We equip the $H^r(\Sigma, GL(k, \mathbb{R}))$ with the distance

$$D_r(A, B) = \sup_X \|A - B\| + h_r(A - B).$$

It is clear the locally constant functions are ∞-Hölder i.e., they are r-Hölder for every $r > 0$, with bounded $h_r(A)$.

Definition 3.4.1. A *local stable holonomy* for the linear cocycles $(T, \mathcal{A})$ is a family of matrices $H^s_{y\leftarrow x} \in GL(k, \mathbb{R})$ defined for all $x \in \Sigma$ with $y \in W^s_{\text{loc}}(x)$ such that

a) $H^s_{x\leftarrow x} = Id$ and $H^s_{z\leftarrow y} o H^s_{y\leftarrow x} = H^s_{z\leftarrow x}$ for any $z, y \in W^s_{\text{loc}}(x)$.

b) $\mathcal{A}(x) \circ H^s_{x\leftarrow y} = H^s_{T(x)\leftarrow T(y)} \circ \mathcal{A}(y)$.

c) $(x, y, v) \mapsto H_{y\leftarrow x}(v)$ is continuous.

Moreover, if $y \in W^u_{\text{loc}}(x)$, then there are analogous properties for $H^u_{x\leftarrow y}$.

According (b) in the above definition, one can extend the definition to the global stable holonomy $H^s_{y\leftarrow x}$ for $y \in W^s(x)$ not necessarily in $W^s_{\text{loc}}(x)$:

$$H^s_{y\leftarrow x} = \mathcal{A}^n(y)^{-1} \circ H^s_{T^n(y)\leftarrow T^n(x)} \circ \mathcal{A}^n(x),$$

where $n \in \mathbb{N}$ is large enough such that $T^n(y) \in W^s_{\text{loc}}(T^n(x))$. One can extend the definition the global unstable holonomy similarly.

Definition 3.4.2. A $r-$Hölder continuous function $\mathcal{A}$ is called *fiber bunched* if for any $x \in \Sigma$,

$$\|\mathcal{A}(x)\|\|\mathcal{A}(x)^{-1}\|\omega^r < 1, \tag{3.4.1.2}$$

where ω is the hyperbolicity constant defining the metric on the base Σ.

We say that the linear cocycle $(T, \mathcal{A})$ is fiber-bunched if its generator $\mathcal{A}$ is fiber-bunched. We denoted by $H^r_b(\Sigma, GL(k, \mathbb{R})))$ the family of r-Hölder-continuous and fiber bunched functions.

The geometric interpretation of the fiber bunching condition is as follows. Let $\mathcal{A} \in H^r_b(\Sigma, GL(k, \mathbb{R}))$. The *projection* cocycle associated to $\mathcal{A}$ and T is the map $\mathbb{P}F : \Sigma \times \mathbb{PR}^k \to \Sigma \times \mathbb{PR}^k$ given by

$$\mathbb{P}F(x, v) := (T(x), \frac{\mathcal{A}(x)v}{\|\mathcal{A}(x)v\|}).$$

We denote by $D\mathcal{A}_v$ the derivative of the action $\mathbb{PR}^k \to \mathbb{PR}^k$ on projective space at all points $v \in \mathbb{PR}^k$. Taking derivative

$$\|D\mathcal{A}_v\| \leq \|\mathcal{A}\|\|\mathcal{A}^{-1}\| \text{ and } \|D\mathcal{A}^{-1}_v\| \leq \|\mathcal{A}\|\|\mathcal{A}^{-1}\|$$

for all $v \in \mathbb{PR}^k$. Therefore, the fiber bunching condition implies that rate of expansion (respectively, contraction) the projective cocycle $\mathbb{P}F$ at every point $x \in \Sigma$ is bounded above by $(\frac{1}{\omega})^r$(respectively, below by ω^r).

The Hölder continuity and the fiber bunched assumption $\mathcal{A} \in H^r_b(\Sigma, GL(k,\mathbb{R}))$ imply the convergence of the *canonical holonomy* $H^{s/u}$ (see [BGMV], [KS]). That means, for any $y \in W^{s/u}_{\text{loc}}(x)$,

$$H^s_{y\leftarrow x} := \lim_{n\to\infty} \mathcal{A}^n(y)^{-1}\mathcal{A}^n(x) \text{ and } H^u_{y\leftarrow x} := \lim_{n\to-\infty} \mathcal{A}^n(y)^{-1}\mathcal{A}^n(x).$$

In addition, when the linear cocycle is fiber bunched, the canonical holonomies vary $r-$Hölder continuisly (see [KS]), i.e., there exists $C > 0$ such that for $y \in W^{s/u}_{\text{loc}}(x)$,

$$\|H^{s/u}_{x\leftarrow y} - I\| \leq Cd(x,y)^r.$$

In this chapter, we will always work with the canonical holonomies for fiber bunched cocycles.

Remark 3.4.3. *Even though the locally constant cocycles are not necessary fiber bunched, the canonical holonomies always exist. Indeed, for every $y \in W^s(x)$ there exist m such that $x_n = y_n$ for all $n \geq m$. Then,*

$$H^s_{x\leftarrow y} = \mathcal{A}^{-1}(x)\cdots\mathcal{A}^{m-1}(x)^{-1}\mathcal{A}^{m-1}(y)\cdots\mathcal{A}(y).$$

In particular $H^s_{x\leftarrow y} = Id$, for all $x \in W^s_{\text{loc}}(y)$. Similarly, we get the existence of the unstable holonomy.

Remark 3.4.4. *If a linear cocycle is not fiber bunched, then it might admit multiple holonomies (see [KS1]).*

3.4.2 Typical cocycles

We are going to discuss typical cocycles. For details, one is referred to [AV], [BV1] and [V].

Suppose that $p \in \Sigma$ is a periodic point of T, we say the $p \neq z \in \Sigma$ is a *homoclinic point* associated to p if it is the intersection of the stable and unstable manifold of p. That is, $z \in W^s(p) \cap W^u(p)$ (see figure 8). The set of homoclinic points of any periodic point is dense in Σ for hyperbolic systems.

We define the *holonomy loop*

$$\psi^z_p := H^s_{z\leftarrow p} \circ H^u_{p\leftarrow z}.$$

Definition 3.4.5. Suppose that $\mathcal{A} : \Sigma \to GL(k,\mathbb{R})$ belongs $H^r_b(\Sigma, GL(k,\mathbb{R}))$. We say that $\mathcal{A}$ is *1-typical* if there exists a periodic point p and a homoclinic point z associated to p such that:

(i) The eigenvalues of $\mathcal{A}^{per(p)}(p)$ have multiplicity 1 and distinct norms. Let $\{v_i\}_{i=1}^k$ be the eigenvectors of $P := \mathcal{A}^{per(p)}(p)$.

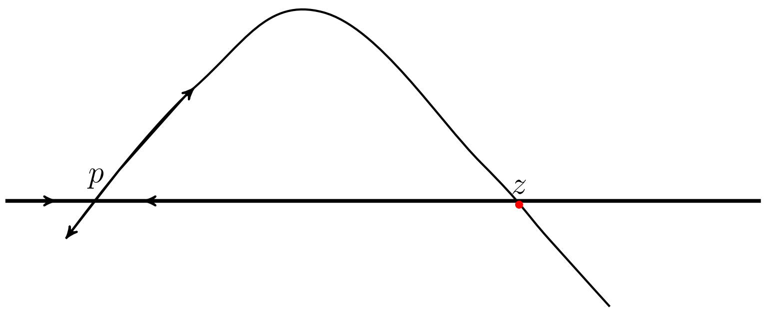


Figure 3: Homoclinic point.

(ii) $\psi_p^z(v_i)$ does not lie in any hyperplane W_j spanned by all eigenvectors of P other than v_i for any $1 \leq i, j \leq k$.

For $k = 2$ this second condition means that $\psi_p^z(v_i)$ does not intersect other lines. See Figure 10 for a 1-typical cocycle in the 2 dimensional case.

We refer to (i) as the *(pinching)* properties and to (ii) as the *(twisting)* properties.

The cocycles generated by $\mathcal{A}^{\wedge t}$, $1 \leq t \leq k$ also admit stable and unstable holonomies, namely $(H^{s/u})^{\wedge t}$.

Definition 3.4.6. Assume that $\mathcal{A}$ is 1-typical. We say $\mathcal{A}$ is *t-typical* for $2 \leq t \leq k-1$, if the points $p, z \in \Sigma$ from Definition 3.4.5 satisfy

(I) $P^{\wedge t}$ satisfies the analogous statement to (i) from Definition 3.4.5 for all t. Let $\{v_{i_1} \wedge ... \wedge v_{i_t}\}_{1 \leq i_1 < ... < i_t \leq k}$ be the eigenvectors of $P^{\wedge t}$.

(II) The induced map $(\psi_p^z)^{\wedge t}$ on $(\mathbb{R}^k)^{\wedge t}$ satisfies the analogous statement to (ii) from Definition 3.4.5.

We say that $\mathcal{A}$ is *typical* if $\mathcal{A}$ is $t-$typical for all $1 \leq t \leq k-1$. We denote by $\mathcal{W} \subset H_b^r(\Sigma, GL(k, \mathbb{R}))$ the set of all typical functions.

Remark 3.4.7. *Above definition for typical cocycles comes from [P] that is a slightly weaker form typical cocycles which was first introduced by Bonatti and Viana [BV1]; Park [P] considered a weaker twisting assumption. We also remark that the difference in the settings of [BV1] and [P] does not make any problems in translating the relevant statements and results from [BV1] to this thesis.*

Remark 3.4.8. *Avila and Viana in [AV] improved the Bonatti and Viana's result by weakening the assumptions: they allowed the number of symbols of Σ to be countably infinite and proved analogous results to [BV1]. They call 1-typical cocycles of*

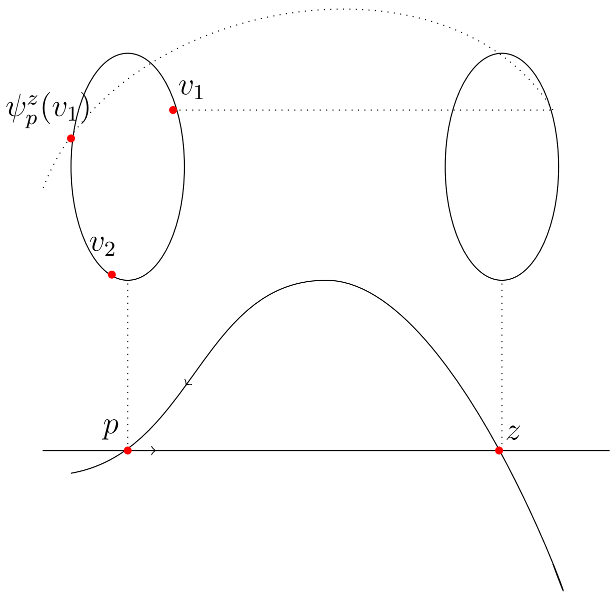


Figure 4: $\psi_p^z(v_1) \neq v_2$

[BV1] by simple cocycles. In comparison to simple cocycles of [AV], our typicality assumption has a weaker twisting assumption, but we still require t-typicality for each $1 \leq t \leq k-1$.

Despite slight variations in the definition of typicality, in all cases, $\mathcal{W}$ is open and dense in $H_b^r(\Sigma, GL(k, \mathbb{R}))$, and its complement has infinite codimension.

Park [P] proved quasi-multiplicativity for typical cocycles $\mathcal{W}$. The approach has its roots in previous work of Feng [F, Proposition 2.8] who showed quasi-multiplicativity for locally constant cocycles under a certain assumption.

Theorem 3.4.9 ([P, Theorem F])**.** *Assume that* $\mathcal{A} \in \mathcal{W}$*. Then* $\mathcal{A}$ *is quasi-multiplicative. Moreover,* $\vec{\Phi}_{\mathcal{A}}$ *is (simultaneously) quasi-multiplicative.*

3.4.3 The continuity of Lyapunov exponents

Throught, $\mathbb{P}F : \Sigma^+ \times \mathbb{PR}^k \to \Sigma^+ \times \mathbb{PR}^k$ is the projective cocycle associated with linear cocycle $F : \Sigma^+ \times \mathbb{R}^k \to \Sigma^+ \times \mathbb{R}^k$ that is generated by $(T, \mathcal{A})$.

We say that a matrix cocycle is *strongly irreducible* when there is no finite family of proper subspaces invariant by $\mathcal{A}(x)$ for μ-almost every x. Furstenberg

[V, Theorem 6.8] showed that the Lyapunov exponent $\chi(\mu, \mathcal{A})$ of F coincides with the integral of the function $\psi : \Sigma^+ \times \mathbb{PR}^k \to \mathbb{R}$,

$$\psi(x, v) = \log \frac{\|\mathcal{A}(x)v\|}{\|v\|}$$

for locally constant cocycles under the strong irreducibility assumption. In other words, he showed that

$$\chi(\mu, \mathcal{A}) = \int \psi d(\mu \times \eta),$$

for any stationary measure η of the associated projective cocycle $\mathbb{P}F$. Therefore, one can easily show that we have the continuity of Lyapunov exponents with respect to $(\mathcal{A}, \mu)$ ([V, Corollary 6.10]) under the strong irreducibility assumption.

Remark 3.4.10. *Bonatti and Viana [BV1], [BGMV] extended the Furstenberg's formula to 1-typical cocycles. Therefore, we have the continuity of Lyapunov exponents for typical cocycles with respect to $(\mathcal{A}, \mu)$, as well.*

Even though discontinuity of Lyapunov exponents is a common features (see [Bo], [Boc1]), there are some results for the continuity of Lyapunov exponents. For instance, Bocker and Viana [BV] proved the continuity of Lyapunov exponents of $2-$dimensional locally constant cocycles under a certain assumption. In order to state the result of Bocker and Viana, we denote by $\triangle_k$ the collection of strictly positive probability vectors in $\mathbb{R}^k$ for $k \geq 2$. We denote by X the full shift space over k symbols. For $p = (p_1, ..., p_k) \in \triangle_k$, let μ be the associated Bernoulli product measure on X.

Theorem 3.4.11 ([BV, Theorem B])**.** *For every $\varepsilon > 0$ there exist $\delta > 0$ and a weak* neighborhood V of μ in the space of probability measures on $GL(2, \mathbb{R})$ such that for every probability measure $\mu' \in V$ whose support is contained in the δ-neighborhood of the support of μ, we have*

$$|\chi(\mu, \mathcal{A}) - \chi(\mu', \mathcal{A}')| < \varepsilon.$$

Avila, Eskin and Viana [AEV] announced recently that Bocker and Viana's result remains true in arbitrary dimensions.

It was conjectured by Viana [V] that Lyapunov exponents are always continuous among $H_b^\alpha(X, GL(2, \mathbb{R}))$-cocycles, and that has been proved by Backes, Brown and Butler [BBB]. In fact, they prove Lyapunov exponents vary continuously on any family of $GL(2, \mathbb{R})$-cocycles with continuous invariant holonomies i.e.,

$$\chi(x, \mathcal{A}_n) \to \chi(x, \mathcal{A}),$$

when $(\mathcal{A}_n, H^{s,n}, H^{u,n}) \to (\mathcal{A}, H^s, H^u)$.

We state the main result of Backes, Brown, and Butler as follows.

Theorem 3.4.12 ([BBB, Theorem 2.8]). *Suppose that $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ is a sequence of 2-dimensional linear cocycles over T converging uniformly with holonomies to a cocycle $\mathcal{A}$ and $\{\mu_n\}_{n\in\mathbb{N}}$ a sequence of fully supported, ergodic, T-invariant probability measures converging to an ergodic, T-invariant measure μ with local product structure and full support. Then*

$$\chi(\mu_n, \mathcal{A}_n) \to \chi(\mu, \mathcal{A}),$$

and,

$$\chi(\mu_n, \mathcal{A}_n^{-1}) \to \chi(\mu, \mathcal{A}^{-1}).$$

That improves Bocker and Viana's result [BV]. Furthermore, Butler [Bu] showed in the following example that the fiber-bunching condition is sharp.

Example 3.4.13. Assume that $T : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ is a shift map. We define a locally constant cocycle (T, A) such that

$$A_0 = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{bmatrix}, \quad A_1 = \begin{bmatrix} \sigma^{-1} & 0 \\ 0 & \sigma \end{bmatrix},$$

where σ is a positive constant greater than 1. We define probability measure ν_p in order to $\nu_p([0]) = p, \nu_p([1]) = 1 - p$, and then Bernoulli measure $\mu_p = \nu_p^{\mathbb{Z}}$. By definition the cocycle (T, A) is fiber bunched if and only if $\sigma^2 < 2^{\alpha}$ [1].

Butler[Bu] shows that for above example if $\sigma^{4p-2} \geq 2^{\alpha}$ for $p \in (\frac{1}{2}, 1)$, then for each neighborhood $\mathcal{U} \subset H^{\alpha}(\{0,1\}^{\mathbb{Z}}, SL(2,\mathbb{R}))$ of A and every $\kappa \in (0, (2p-1)\log\sigma]$, there is a locally constant cocycle $\mathcal{B} \in \mathcal{U}$ such that $\chi(x, \mathcal{B}) = \kappa$. In particular, A is a discontinuity point for Lyapunov exponents in $H^{\alpha}(\{0,1\}^{\mathbb{Z}}, SL(2,\mathbb{R}))$. So, this example shows that we have discontinuity of Lyapunov exponents near fiber bunched cocycles.

The inequality $\sigma^{4p-2} \geq 2^{\alpha}$ comes from the following observation

$$\lim_{n\to\infty} \log(\|A^n(x)\|\|A^n(x)^{-1}\|) = (\chi(\mu_p, A) - \chi(\mu_p, A^{-1})) = (4p-2)\log\sigma,$$

for μ_p almost every $x \in \{0,1\}^{\mathbb{Z}}$.

Lyapunov exponents are T-invariant maps, thus when μ is ergodic they are constant μ-almost everywhere. In that case, we denote them as $\chi_i(\mathcal{A})$ for $i = 1, .., k$.

For $\vec{\alpha} \in \mathbb{R}^k$, the *Lyapunov spectrum* of linear cocycles is defined as:

$$\vec{L} := \{\vec{\alpha}, \ \exists x \in \Sigma \text{ such that } \chi_l(\mathcal{A}) = \alpha_l\},$$

for $1 \leq l \leq k$.

[1] Σ is equipped by a norm d that is, for all $x \neq y$, $d(x,y) = 2^{-N(x,y)}$, where $N(x,y) = \min\{n, x_n \neq y_n\}$.

Theorem 3.4.14 ([P, Theorem D]). *Let $\mathcal{A} \in \mathcal{W}$. Then $\vec{L}$ is a convex and closed subset of $\mathbb{R}^k$.*

We use Theorem 3.4.12 to show that the Lyapunov spectrum of fiber bunched cocycles is a closed and convex set.

Corollary 3.4.15. *Let $\mathcal{A} \in H_b^r(X, GL(2,\mathbb{R})))$. Then $\vec{L}$ is a convex and closed subset of $\mathbb{R}^2$.*

Proof. Since $\mathcal{W}$ is open and dense, for every $\mathcal{A} \in H_b^r(X, GL(2,\mathbb{R})))$ there is a $\mathcal{A}_k \in \mathcal{W}$ such that $\mathcal{A}_k \to \mathcal{A}$.

By Theorem 3.4.12,

$$\chi_i(\mathcal{A}_k) \to \chi_i(\mathcal{A})$$

for $i = 1, 2$. By Theorem 3.4.14, $\vec{L}$ is a closed and convex subset of $\mathbb{R}^2$. $\square$

3.4.4 Thermodynamic formalism for linear cocycles

In this subsection we will present what is known for linear cocycles.

Feng and Käenmäki [FK] extended the Bowen's result, who proved the uniqueness of equilibrium state for additive potentials under certain assumptions, for subadditive potentials $t\Phi$ on a locally constant cocycle under the assumption that the matrices in $\mathcal{A}$ do not preserve a common proper subspace of $\mathbb{R}^k$ (i.e. $(T, \mathcal{A})$ is irreducible).

Consider Theorem 3.4.9, the following theorem shows that we have the Feng and Käenmäki's result for typical cocycles.

Theorem 3.4.16. *Let $\mathcal{A} \in \mathcal{W}$ be typical. Assume that $\vec{\Phi}_{\mathcal{A}}$ is (simultaneously) quasi-multiplicative and $\vec{t} \in \mathbb{R}^k_+$. Then $P_{\vec{\Phi}_{\mathcal{A}}}(\vec{t})$ has a unique equilibrium state $\mu_{\vec{t}}$ for the subadditive potential $\vec{t}.\vec{\Phi}_{\mathcal{A}}$. Furthermore, $\mu_{\vec{t}}$ has the following Gibbs property: There exists $C \geq 1$ such that for any $n \in \mathbb{N}$, $[J] \in \mathcal{L}(n)$, we have*

$$C^{-1} \leq \frac{\mu_{\vec{t}}([J])}{e^{-nP_{\vec{\Phi}_{\mathcal{A}}}(\vec{t}) + \vec{t}.\ \vec{\Phi}_{\mathcal{A}}(x)}} \leq C, \tag{3.4.4.1}$$

for any $x \in [J]$. Furthermore, $P_{\vec{\Phi}_{\mathcal{A}}}(.)$ is differentiable on $\mathbb{R}^k_+$ and $\nabla P_{\vec{\Phi}_{\mathcal{A}}}(\vec{t}) = \chi(\mu_{\vec{t}}, \vec{\Phi}_{\mathcal{A}})$.

Proof. It is easily follows from Lemma 2.5.2 and [P, Proposition 3.9]. $\square$

Park [P] uses the quasi-multiplicative property $\mathcal{A} \in \mathcal{W}$ to show the continuity of the topological pressure which it states in the following theorem. We remark that we prove that for $\mathcal{A} \in H_b^\alpha(\Sigma, GL(2,\mathbb{R}))$ in the next section.

Theorem 3.4.17 ([P, Theorem B]). *The map $(s, \mathcal{A}) \to P_{\tilde{\Phi}_{\mathcal{A}}}(s)$ is continuous on $[0, \infty) \times \mathcal{W}$.*

Theorem 3.4.18. *Assume that $h_{top}(T) < \infty$, and $\alpha(\mathcal{A}) < \infty$. If $\mathcal{A} \in \mathcal{W}$, then $P_{\Phi_{\mathcal{A}}}()$ is a real continuous convex function on $\mathbb{R}$. Moreover, $\alpha(\mathcal{A})$ exists and it is equal $P'_{\Phi_{\mathcal{A}}}(-\infty) := \lim_{t\to-\infty} \frac{P_{\Phi_{\mathcal{A}}}(t)}{t}$. Similarly, $P_{\vec{\Phi}_{\mathcal{A}}}$ is a real continuous convex function on $\mathbb{R}^k$. Furthermore,*

$$\begin{aligned}\vec{\alpha}(\mathcal{A}) &:= \min\{\alpha_i, \vec{\alpha} \in \vec{L}\} \\ &= \lim_{\vec{t}\to-\infty} \frac{P_{\vec{\Phi}_{\mathcal{A}}}(\vec{t})}{\vec{t}}.\end{aligned}$$

Proof. See [F, Lemmas 2.2 and 2.3]. We remark that although [F, Lemmas 2.2 and 2.3] only deal with locally constant cocycles, the proof given there works for our theorem under slightly modification. Indeed, Feng uses the quasi-multiplicative properties to prove the lemmas. Since $\mathcal{A} \in \mathcal{W}$, $\vec{\Phi}_{\mathcal{A}}$ is (simultaneously) quasi-multiplicative by Theorem 3.4.9. □

3.5 The results

3.5.1 The proof of Theorem 1.2.2

In this subection we discuss multifractal formalism of typical cocycles. Our motivation for studying of the multifractal formalism associated to certain iterated function systems with overlaps. For instance, the Hausdorff dimension of level sets has been calculated for 2-dimension-planar affine iterated function systems satisfying strong irreducibility and the strong open set condition by B. Bárány, T. Jordan, A. Käenmäki, and M. Rams [BJKR]. In the additive potential setting, the Lyapunov exponents are equal the Birkhoff averages. In this case, the restricted varitional principle, topological entropy, and Hausdorff dimension level set has been studied by a lot of authors (see [C]).

Theorem 3.5.1. *Let $\mathcal{A} \in \mathcal{W}$. Suppose that $P_{\vec{\Phi}_{\mathcal{A}}}(\vec{q}) \in \mathbb{R}$ for each $\vec{q} \in \mathbb{R}^k$. Then for $\vec{\alpha} \in \vec{L}$,*

$$h_{top}(T, E(\vec{\alpha})) = \inf\{P_{\vec{\Phi}_{\mathcal{A}}}(\vec{q}) - \vec{\alpha}.\vec{q} : \ \vec{q} \in \mathbb{R}^k\}. \tag{3.5.1.1}$$

Proof. One can find the proof in [FH, Theorem 4.10] and [F, Theorem 1.1]. We remark that although Feng only deals with the locally constant cocycles, the proof given there works under slightly modification. □

Theorem 3.5.2. *Assume that $T : \Sigma \to \Sigma$ is a topologically mixing subshift of finite type on the compact metric space Σ. Suppose that $\mathcal{A} : X \to GL(k, \mathbb{R})$ belongs to typical functions $\mathcal{W}$. Assume that ω is the range of the map from $\mathcal{M}(\Sigma, T)$ to $\mathbb{R}^k$*

$$\mu \mapsto (\chi_1(\mu, \mathcal{A}), \chi_2(\mu, \mathcal{A}), ..., \chi_k(\mu, \mathcal{A})).$$

We define

$$\boldsymbol{h}(\vec{\alpha}) := \sup\{h_\mu(T) : \mu \in \mathcal{M}(\Sigma, T), \chi_i(\mu, \mathcal{A}) = \alpha_i\},$$

where $\vec{\alpha} \in \omega$. Then,

$$\boldsymbol{h}(\vec{\alpha}) = \inf\{P_{\vec{\Phi}_\mathcal{A}}(\vec{q}) - \vec{\alpha}.\vec{q} : \vec{q} \in \mathbb{R}^k\}.$$

Proof. Fix $\vec{\alpha} = (\alpha_1, ..., \alpha_k) \in \omega$. For $\mu \in \mathcal{M}(\Sigma, T)$, we define

$$V_{\vec{\alpha}}(\mu) := (\chi_1(\mu, \mathcal{A}) - \alpha_1, ..., \chi_k(\mu, \mathcal{A}) - \alpha_k).$$

It is easy to see that there is $\mu^{'} \in \mathcal{M}(\Sigma, T)$ such that $V_{\vec{\alpha}}(\mu^{'}) = \vec{0}$.

We write

$$A = \{V_{\vec{\alpha}}(\mu) : \mu \in \mathcal{M}(\Sigma, T)\}.$$

$V_{\vec{\alpha}}(.)$ is a continuous affine function on $\mathcal{M}(X, T)$ (see remark 3.4.10). Therefore, A is a convex compact set in $\mathbb{R}^k$.

We define $g : A \to \mathbb{R}$ by

$$g(\vec{t}) = \sup\{h_\mu(T) : \mu \in \mathcal{M}(\Sigma, T),\ V_{\vec{\alpha}}(\mu) = \vec{t}\}.$$

It is easy to see that g is a concave and upper semi continuous function on A. We have $\mathbf{h}(\vec{\alpha}) = g(0)$. We define

$$W(\vec{q}) := \sup\{g(\vec{t}) + \vec{q}.\vec{t} : \vec{t} \in \mathbb{R}^k\},$$

for all $\vec{q} \in A$. Then, we have

$$g(\vec{t}) = \inf\{W(\vec{q}) - \vec{q}.\vec{t} : \vec{q} \in \mathbb{R}^k\}$$

for all $\vec{t} \in A$, by Corollary 3.3.2. Hence, we have

$$\mathbf{h}(\vec{\alpha}) = g(\vec{0}) = \inf\{W(\vec{q}) : \vec{q} \in \mathbb{R}^k\}.$$

By Theorem 3.4.18, $P_{\vec{\Phi}_\mathcal{A}}(\vec{q})$ is a convex function on $\mathbb{R}^k$. Then, by variational principle

$$W(\vec{q}) = P_{\vec{\Phi}_\mathcal{A}}(\vec{q}) - \vec{\alpha}.\vec{q}.$$

□

Remark 3.5.3. *In the locally constant cocycles case, Theorem 3.5.2 is true under the strong irreducibility assumption, which means we do not need the pinching assumption in this case.*

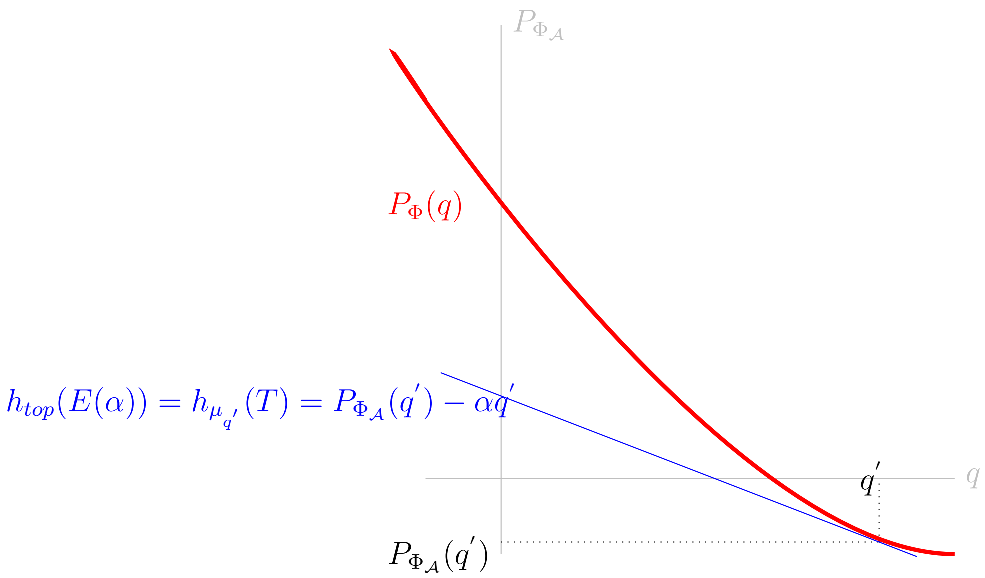


Figure 5: $P_{\Phi_{\mathcal{A}}}(.)$ is a convex function for $q \in \mathbb{R}$. The blue line is tangent to $P_{\Phi_{\mathcal{A}}}(.)$ at q' with slope $\alpha = P'_{\Phi_{\mathcal{A}}}(q')$.

3.5.2 The proof of Theorem 1.2.1

In this subsection we are going to show that the closure of the set where the entropy spectrum is positive is equal the Lyapunov spectrum for typical cocycles. This result is first attempt to characterize Lyapunov spectrum as a set of positive entropy spectrum. The main input of our argument will be the fact that the topological pressure is convex for typical cocycles. Then, we can show the concavity of the entropy spectrum of Lyapunov exponents by Theorem 1.2.2.

We recall that $T : \Sigma \to \Sigma$ is a topologically mixing subshift of finite type and $\mathcal{A} : \Sigma \to GL(k, \mathbb{R})$ is a Hölder continuous function. We always assume that we have more than a matrix, otherwise it might be topological entropy equal zero. The interior and topological closure of a set A is denoted by $\mathring{A}$ and $\overline{A}$.

Lemma 3.5.4. *Let $\mathcal{A} \in \mathcal{W}$. Then, $h_{top}(E(\alpha))$ is concave on the convex set $\mathring{L}$.*

Proof. The topological pressure $P_{\Phi_{\mathcal{A}}}(.)$ is convex by Theorem 3.4.18. Moreover, according to Theorems 3.5.1 and 3.5.2 we have

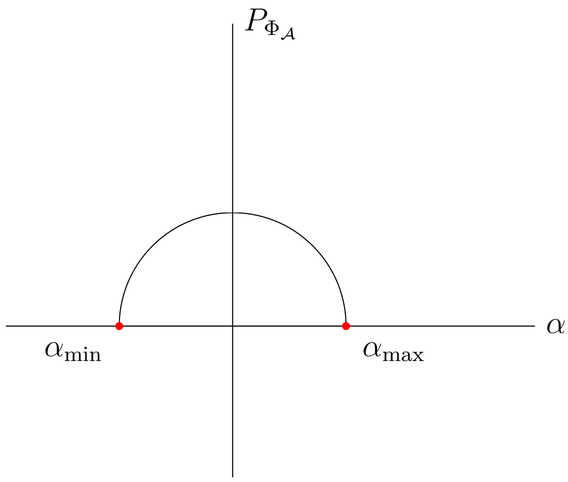
$$\begin{aligned} h_{top}(E(\alpha)) &= \inf_{t\in\mathbb{R}}\{P_{\Phi_{\mathcal{A}}}(t) - \alpha t\} \\ &= \sup\{h_\mu(T): \ \mu \in \mathcal{M}(\Sigma, T), \ \chi(\mu, \mathcal{A}) = \alpha\}. \end{aligned}$$

Hence, it is concave. □

Theorem 3.5.5. *For $\alpha \in \mathring{L}$, $h_{top}(E(\alpha)) > 0$.*

Proof. By Theorem 3.5.4 $h_{top}(E(\alpha))$ is concave. Thus, $h_{top}(E(\alpha)) > 0$. Indeed, a concave function with a maximum in the interior of the domain is non-decreasing to the left of the maximum and non-increasing to the right of the maximum. □

Remark 3.5.6. *Entropy spectrum at boundary of Lyapunov spectrum is not necessarily positive. In fact, there is a conjecture, which is known as Meta conjecture, that says that under generic assumptions the entropy spectrum at boundary of Lyapunov spectrum is zero (which would mean that $h_{top}(E(\beta(\mathcal{A})) = h_{top}(E(\alpha(\mathcal{A})) = 0$); this phenomenon is often referred to as ergodic optimization of Lyapunov exponents, see for example [Bo]. In the additive potential case, instead, this phenomenon is often referred to as ergodic optimization of Birkhoff averages, see for example [J].*



Theorem 3.5.7. $\overline{\{\alpha \in \mathbb{R}, \ h_{top}((E(\alpha)) > 0\}} = L$.

Proof. That is direct consequence Theorem 3.5.5. □

Park [P] proved Theorem 3.4.14 for higher dimensional case. That means, $\vec{L}$ is closed and convex. So, we can obtain the following generalization of Theorem 3.5.7 to the Lyapunov spectrum of of all Lyapunov exponents.

Theorem 3.5.8. $\overline{\{\vec{\alpha} \in \mathbb{R}^k,\ \ h_{top}(E(\vec{\alpha})) > 0\}} = \vec{L}.$

We remark that the concavity of a function defined on a convex set implies the continuity of the function in the interior, and that the continuity of the entropy under the change of the Lyapunov exponents implies the continuity of the Lyapunov spectrum.

3.5.3 The proof of Theorem 1.2.3

We start with the key Proposition 3.3.7 which tells us that the subderivative of the topological pressure for a subadditive potential is equal to the Lyapunov exponent of the equilibrium measure. Let (X, T) be a topological dynamical systems such that the entropy map $\mu \mapsto h_\mu(T)$ is upper semi-continuous and $h_{top}(T) < \infty$. Suppose that $\Phi = \{\log \Phi_n\}_{n=1}^{\infty}$ is a subadditive potential on the compact metric space X.

Theorem 3.5.9. *For each $t > 0$, the family of equilibrium measures (μ_t), has a weak* accumulation point as $t \to \infty$. Any such accumulation point μ is a Lyapunov maximizing measure for Φ. Moreover,*

$$\chi(\mu, \Phi) = \lim_{t \to \infty} \chi(\mu_t, \Phi).$$

Moreover, the maximal Lyapunov exponent can be approximated by Lyapunov exponents of equilibrium measures.

Proof. It is obvious that (μ_t) has at least one accumulation point, let us call it μ. By Theorem 3.3.3, $P(t)$ is convex , then we have $\partial P(t) = \{\chi(\mu_t, \Phi)\}$ by Proposition 3.3.7. Moreover, since $P(t)$ is convex for $t > 0$, $t \mapsto \chi(\mu_t, \Phi)$ is non-decreasing and bounded above[2].
It follows that

$$\lim_{t \to \infty} \partial P(t) = \lim_{t \to \infty} \chi(\mu_t, \Phi) \text{ exsits and is finite.}$$

Since Lyapunov exponents are upper semi continuous,

$$\lim_{t \to \infty} \chi(\mu_t, \Phi) \leq \chi(\mu, \Phi).$$

By the definition of $\mathrm{Eq}(\Phi, t)$,

$$\chi(\mu_t, \Phi) + \frac{h_{\mu_t}(T)}{t} \geq \chi(\mu, \Phi) + \frac{h_\mu(T)}{t}. \tag{3.5.3.1}$$

[2]It follows from subadditivity.

Since the topological dynamical systems (X,T) has finite topological entropy, so when $t \to \infty$, (3.5.3.1) implies

$$\lim_{t\to\infty} \chi(\mu_t, \Phi) \geq \chi(\mu, \Phi).$$

Now, we shall show that μ is a Lyapunov maximizing measure.

By contradiction, let us assume that there exists ν with $\chi(\nu,\Phi) - \chi(\mu,\Phi) = \kappa > 0$. One can define the affine map $T_\nu : \mathbb{R}_+ \to \mathbb{R}$ by $T_\nu(t) = h_\nu(T) + t\chi(\nu,\Phi)$. We know that $t \mapsto \chi(\mu_t, \Phi)$ is a function which increases to its limit $\chi(\mu,\Phi)$, so

$$\begin{aligned} &\chi(\mu,\Phi) \geq \chi(\mu_{t_*}, \Phi) = \partial^e P(t_*), \text{where } t_* = t_- \text{ or } t_+, \\ &\text{and } T_\nu^{'}(t) = \chi(\nu,\Phi) = \chi(\mu,\Phi) + \kappa \geq \partial^e P(t_*) + \kappa. \end{aligned}$$

Consequently, $h_\nu(T) + t\chi(\nu,\Phi) > P(t)$ for all sufficiently large $t > 0$, that contradicts by our assumption. So, μ is Lyapunov maximizing measure.

Moreover, our proof implies that $\beta(\Phi)$ can be approximated by Lyapunov exponents of equilibrium measures of a subadditive potential $t\Phi$. □

Lemma 3.5.10. *The maps $t \mapsto h_{\mu_t}(T)$ and $t \mapsto P(t\Phi - t\beta(\Phi))$ are non-increasing and bounded below on the interval $(0,\infty)$. Moreover, we have*

$$\lim_{t\to\infty} h_{\mu_t}(T) = \lim_{t\to\infty} P(t\Phi - t\beta(\Phi)) \geq \sup_{\nu\in\mathcal{M}_{\max}(\Phi)} h_\nu(T).$$

Proof. The map $t \mapsto P(t\Phi - t\beta(\Phi))$ is convex. By definition of $\beta(\Phi)$,

$$\chi(\mu_t,\Phi) \leq \beta(\Phi) \text{ for all } \mu_t \in \mathrm{Eq}(t).$$

We assume that $P(t) = P(t\Phi)$. By the definition of the topological pressure, $P(t\Phi - t\beta(\Phi)) = P(t\Phi) - t\beta(\Phi)$. Then,

$$\partial^e P(t_*\Phi - t_*\beta(\Phi)) = \partial^e P(t_*\Phi) - \beta(\Phi) = \chi(\mu_{t_*},\Phi) - \beta(\Phi) \leq 0,$$

where $t_* = t_-$ or t_+. Thus, $P(t\Phi - t\beta(\Phi))$ is non-increasing by Theorem 2.3.1. We are going to show that $t \mapsto h_{\mu_t}(T)$ is non-increasing. Since μ_t is an equilibrium measure,

$$h_{\mu_{t_*}}(T) = P(t) - t\partial^e P(t_*).$$

For $0 < x < y$ we have

$$\partial^e P(x_*) \leq \frac{P(y) - P(x)}{y - x} \leq \partial^e P(y_*),$$

so $P(y) - P(x) \leq y\partial^e P(y_*) - x\partial^e P(y_*) \leq y\partial^e P(y_*) - x\partial^e P(x_*)$, and then

$$P(y) - y\partial^e P(y_*) \leq P(x) - x\partial^e P(x_*).$$

Since $t \mapsto h_{\mu_t}(T)$ and $t \mapsto P(t\Phi - t\beta(\Phi)) \geq 0$ are non-increasing and non-negative, we conclude that $\lim_{t\to\infty} h_{\mu_t}(T)$ and $\lim_{t\to\infty} P(t\Phi - t\beta(\Phi))$ both exist. This implies that the limit

$$\lim_{t\to\infty} t\partial^e P(t) - t\beta(\Phi) = \lim_{t\to\infty}(P(t\Phi - t\beta(\Phi)) - h_{\mu_t}(T))$$

exists. Then,

$$\lim_{t\to\infty} h_{\mu_t}(T) = \lim_{t\to\infty} P(t\Phi - t\beta(\Phi)).$$

Last part follows from the variational principle. □

Lemma 3.5.11. $\mathcal{M}_{\max}(\Phi)$ *is compact, convex and nonempty, and its extreme points are precisely its ergodic elements.*

Proof. See [M3, Appendix A]. □

Theorem 3.5.12. $h_\mu(T) = \lim_{t\to\infty} h_{\mu_t}(T) = \max\{h_\nu(T), \nu \in \mathcal{M}_{\max}(\Phi)\}$.

Proof. By Theorem 3.5.9 and Lemmas 3.5.10 and 3.5.11,

$$h_\mu(T) \leq \max_{\nu\in\mathcal{M}_{max}(\Phi)} h_\nu(T) \leq \lim_{t\to\infty} h_{\mu_t}(T),$$

the reverse inequality follows from upper semi continuity of entropy. □

Remark 3.5.13. *Let $(T, \mathcal{A})$ be a locally constant cocycle. Then, one can prove the above results for Gibbs measures under the assumption that $(T, \mathcal{A})$ is irreducible (see [FK]). Moreover, if $T : X \to X$ is a mixing subshift of finite type and $\mathcal{A} : X \to GL(k, \mathbb{R})$ is a Hölder continuous function, then one can prove the above results for Gibbs measures under the generic assumption (typical cocycles).*

3.5.4 Approximation of the maximal Lyapunov exponent

In this subsection, we consider locally constant cocycles and we prove the maximal Lyapunov exponent can be approximated by Lyapunov exponents of periodic trajectories.

Kalinin and Sadovskaya [KS] proved that if a homeomorphism T satisfies the Anosov closing property, and $\mathcal{A} : X \to GL(k, \mathbb{R})$ is a Hölder continuous Banach cocycle, then the maximal Lyapunov exponent can be approximated by Lyapunov exponents of measures supported on periodic orbits. In general, Kalinin [Ka] shows that for a Hölder continuous map $\mathcal{A} : X \to GL(k, \mathbb{R})$, Lyapunov exponents can be approximated by Lyapunov exponents of measures supported on periodic orbits under an assumption slightly stronger than the Anosov closing property. Our approach differs from them. We use the continuity of Lyapunov exponents (Theorem 3.4.11), the Anosov closing property and Theorem 3.3.10 for the proof.

Let $(T, \mathcal{A})$ be the locally constant cocycle where $\mathcal{A} : X \to GL(2, \mathbb{R})$. We denote $\phi_n := \|\mathcal{A}_n\|$.

Theorem 3.5.14. *Suppose that T satisfies the Anosov closing property. Then the maximal Lyapunov exponent $\beta(\Phi)$ can be approximated by Lyapunov exponents of measures supported on periodic orbits.*

Proof. Let μ be an ergodic maximizing measure, that is $\beta(\Phi) = \chi(\mu, \Phi)$.

Let x be a generic point for μ. Then there exists $\mu_{n,x} := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{T^j(x)}$, where δ_x is the Dirac measure at the point x, so that $\mu_{x,n} \to \mu$. According to Theorem 3.4.12, and (3.2.1)

$$\lim_{n\to\infty} \frac{1}{n} \log \phi_n(x) = \chi(\mu, \Phi).$$

Let $p \in X$ be a periodic point associated to ε, C, δ and

$$\{x, T(x), ..., T^{n-1}(x)\}$$

by the Anosov closing property. Denote by $\mu_p := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{T^j(p)}$ the ergodic $T-$invariant measure supported on the orbit of p.

Lemma 3.5.15. $\mu_p \to \mu$ *in weak* topology.*

Proof. We will use the Anosov closing property .

Assume that (f_m) is a sequence of continuous functions. The periodic orbit p has length n and is close to the initial segment of the orbit of x. Since the f_m's are continuous, the average of f_m along the periodic orbit is very close to the average of f_m along the first n iterates of x, and that is very close to $\int f_m d\mu$ by the genericity condition. Then, for n large enough, we get longer and longer periodic orbits, approaching x more closely, and we obtain the convergence of the measures to μ. ☐

We now use Lemma 4.3.4 to finish the proof. By the Anosov closing property, periodic point p is close to x, with iterates also close to the iterates of x. Therefore, Theorem 3.4.11 implies for every $\varepsilon > 0$

$$\chi(p, \Phi) = \lim_{n\to\infty} \frac{1}{n} \log \phi_n(p) = \lim_{n\to\infty} \frac{1}{n} \log \phi_n(x) + \varepsilon. \tag{3.5.4.1}$$

Applying Lemma 4.3.4, Theorem 3.4.12 and (3.5.4.1), we obtain

$$\chi(p, \Phi) = \lim_{n\to\infty} \frac{1}{n} \log \phi_n(p) = \chi(\mu, \Phi) + \varepsilon = \beta(\Phi) + \varepsilon.$$

☐

Remark 3.5.16. *One can prove the above theorem for 2-dimensional fiber bunched linear cocycles by using Theorem [BBB].*

Remark 3.5.17. *Avila, Eskin and Viana [AEV] announced recently that the Theorem 3.4.11 remains true in arbitrary dimensions. By their result, the proof given for Theorem 3.4.11 works for arbitrary dimensions.*

Remark 3.5.18. *Morris [M2] showed that the speed of convergence of Theorem 3.5.14 is always superpolynomial for locally constant cocycles. Moreover, Bochi and Garibaldi [BG] show that it is true for general cocycles under certain assumptions.*

3.5.5 The proof of Theorem 1.2.4

In this subsection, we will discuss the continuity of the entropy spectrum of Lyapunov exponents, that is, the topological entropy of level sets of points with a common given Lyapunov exponent. In the locally constant cocycles case, Lemma 3.5.19 follows from Feng and Shmerkin's paper [FS]; see [FS, Proposition 5.3].

Lemma 3.5.19. *Assume $\mathcal{A}_k, \mathcal{A} \in \mathcal{W}$ with $\mathcal{A}_k \to \mathcal{A}$. For $t_k, t > 0$, let $t_k \to t$. Suppose that α_{t_k} and α_t are the derivatives of $P_{\Phi_{\mathcal{A}_k}}()$ and $P_{\Phi_{\mathcal{A}}}()$ at t_k and t, respectively. Then,*

$$\lim_{k\to\infty} h_{top}(E_{\mathcal{A}_k}(\alpha_{t_k})) = h_{top}(E_{\mathcal{A}}(\alpha_t)).$$

Proof. According to Theorem 3.4.16 $P_{\Phi_{\mathcal{A}}}()$ is differentiable for any $t > 0$ and there is a unique equilibrium measure μ_t for the subadditive potential $t\Phi_{\mathcal{A}}$. Therefore, we have

$$h_{top}(E_{\mathcal{A}}(\alpha_t)) = h_{\mu_t}(T),$$

where $P'_{\Phi_{\mathcal{A}}}(t) = \alpha_t$, by Theorem 3.3.9.

Taking into account the observation above, to prove the theorem it is enough to show that $h_{\mu_{t_k}}(T) \to h_{\mu_t}(T)$ for proving the theorem.

By the definition of $\mathrm{Eq}(\Phi_{\mathcal{A}_k}, t_k)$,

$$P_{\Phi_{\mathcal{A}_k}}(t_k) = h_{\mu_{t_k}}(T) + t_k\chi(\mu_{t_k}, \mathcal{A}_k).$$

Notice that the Lyapunov exponents are upper semi-continuous. Moreover, the topologically mixing subshift of finite type $T : \Sigma \to \Sigma$ implies upper semi-continuity of the entropy map $\mu \mapsto h_\mu(T)$. Now, we conclude from above observations and Theorem 3.4.17,

$$\begin{aligned} P_{\Phi_{\mathcal{A}}}(t) &= \lim_{k\to\infty} P_{\Phi_{\mathcal{A}_k}}(t_k) \\ &= \lim_{k\to\infty} h_{\mu_{t_k}}(T) + t_k\chi(\mu_{t_k}, \mathcal{A}_k) \\ &\le h_{\mu_t}(T) + t\chi(\mu_t, \mathcal{A}). \end{aligned}$$

This shows $\mu_t \in \mathrm{Eq}(\Phi_{\mathcal{A}}, t)$ and $\mu_{t_k} \to \mu_t$ [3]. Moreover, we have equality in the above, which gives the claim. Furthermore, it shows the continuity of Lyapunov exponents of equilibrium measures. $\square$

We use Theorems 3.5.9 and 3.5.12 to prove the following theorem.

Theorem 3.5.20. *Suppose that $\mathcal{A} \in \mathcal{W}$. If $\alpha_t = P'_{\Phi_{\mathcal{A}}}(t)$[4] for $t > 0$. Then,*

$$h_{top}(E(\alpha_t)) \to h_{top}(E(\beta(\mathcal{A})) \ \ \textit{when} \ \ t \to \infty.$$

Proof. Since $\mathcal{A} \in \mathcal{W}$, Theorem 3.4.16 implies that there is a unique equilibrium state μ_t for the subadditive potential $t\Phi_{\mathcal{A}}$ such that

$$\chi(\mu_t, \mathcal{A}) = \alpha_t = P'_{\Phi_{\mathcal{A}}}(t).$$

By Theorem (3.3.9),

$$h_{top}(E(\alpha_t)) = h_{\mu_t}(T).$$

We know that

$$h_{top}(E(\beta(\mathcal{A})) = \sup\{h_\mu(T), \ \ \mu \in \mathcal{M}(\Sigma, T), \ \ \chi(\mu, \mathcal{A}) = \beta(\mathcal{A})\}$$

by Theorem 3.3.6. Therefore, we only need to show that

$$h_{\mu_t}(T) \to \sup\{h_\mu(T), \ \ \mu \in \mathcal{M}(\Sigma, T), \ \ \chi(\mu, \mathcal{A}) = \beta(\mathcal{A})\}.$$

That follows from Theorem 3.5.12. $\square$

Theorem 3.5.21. *Suppose $\mathcal{A}_l, \mathcal{A} \in \mathcal{W}$ with $\mathcal{A}_l \to A$, and $\vec{t_l}, \vec{t} \in \mathbb{R}^k_+$ such that $t_l \to t$. Assume $\vec{\alpha_{t_l}} = \nabla P_{\vec{\Phi}_{\mathcal{A}_l}}(\vec{t_l})$ and $\vec{\alpha_t} = \nabla P_{\vec{\Phi}_{\mathcal{A}}}(\vec{t})$. Then,*

$$\lim_{l\to\infty} h_{top}(E(\vec{\alpha_{t_l}})) = h_{top}(E(\vec{\alpha_t})).$$

Moreover,

$$h_{top}(E(\vec{\alpha_t})) \to h_{top}(E(\vec{\beta}(\vec{\Phi}_{\mathcal{A}})) \ \ \textit{when} \ \ t \to \infty.$$

Proof. The proof is similar to Theorems 3.5.20 and 3.5.12 and is omitted. $\square$

[3] By weak-* compactness μ_{t_k} has a accumulation point, let us call μ_t. According the above observation μ_t is an equilibrium measure for $t\Phi_{\mathcal{A}}$. Then uniqueness of equilibrium measure implies the limit.

[4] We remind the reader that $P(.)$ is differentiable for $\mathcal{A} \in \mathcal{W}$ according to Theorem 3.4.16

3.5.6 The proof of Theorem 1.2.5

In this subsection, we are going to prove the continuity of the lower joint spectral radius for derivative cocycles under certain assumptions. This kind of result is known by Bochi and Morris [BM] under 1-domination assumption for locally constant cocycles. Breuillard and Sert [BS] extended their result to the joint spectrum of locally constant cocycles. Moreover, they gave a counterexample [BS, Example 4.13] that shows that we have discontinuity the lower joint spectral for typical cocycles. Even though, we have a lot of results for the upper spectral radius, we have few result about the lower spectral radius, which shows that working on the later case is much harder than the former case.

Assume that $T : X \to X$ is a diffeomorphism on a compact invariant set X. Let $V \oplus W$ be a splitting of the tangent bundle over X that is invariant by the tangent map DT. In this case, if vectors in V are uniformly contracted by DT and vectors in W are uniformly expanded, then the splitting is called *hyperbolic*. The more general notion is the *dominated splitting*, if at each point all vectors in V are more contracted than all vectors in W. Domination could also be called uniform projective hyperbolicity. Indeed, domination is equivalent to V being hyperbolic repeller and W being hyperbolic attractor in the projective bundle.

In the linear cocycles case, we are interested in bundles of the form $X \times \mathbb{R}^k$, where the linear cocycles is generated by $(T, \mathcal{A})$. Bochi and Gourmelon [BGO] showed that a cocycle admits a dominated splitting $V \oplus W$ with $\dim V = k$ if and only if when $n \to \infty$, the ratio between the $k-th$ and $(k+1)-th$ singular values of the matrices of the $n-th$ iterate increase uniformly exponentially. In fact, they extended the Yoccoz's result [Y] that was proved for 2-dimensional vector bundles.

Definition 3.5.22. We say that $\mathcal{A}$ is *i-dominated* if there exist constants $C > 1$, $0 < \tau < 1$ such that

$$\frac{\sigma_{i+1}(\mathcal{A}^n(x))}{\sigma_i(\mathcal{A}^n(x))} \leq C\tau^n, \ \ \forall n \in \mathbb{N}, x \in X.$$

According to the multilinear algebra properties, where $\mathcal{A}$ is $i-$dominated if and only if $\mathcal{A}^{\wedge i}$ is $1-$dominated.

Let (X, T) be a TDS. We say that $\mathcal{A} : X \to GL(k, \mathbb{R})$ is *almost multiplicative* if there is a constant $C > 0$ such that

$$||\mathcal{A}^{m+n}(x)|| \geq C||\mathcal{A}^m(x)||||\mathcal{A}^n(T^m(x))|| \ \forall x \in X, m.n \in \mathbb{N}.$$

We note that since clearly $||\mathcal{A}^{m+n}(x)|| \leq ||\mathcal{A}^m(x)||||\mathcal{A}^n(T^m(x))||$ for all $x \in X, m.n \in \mathbb{N}$, the condition of almost multiplicativity of $\mathcal{A}$ is equivalent to the statement that $\Phi_{\mathcal{A}}$ is almost additive.

Proposition 3.5.23. *Let X be a compact manifold, and let $A : X \to GL(k, \mathbb{R})$ be a matrix cocycle over a TDS (T, X). Let $(C_x)_{x\in X}$ be an invariant cone field on X. Then, there exists $\kappa > 0$ such that for every $m, n > 0$ and for every $x \in X$ we have*

$$||\mathcal{A}^{m+n}(x)|| \geq \kappa ||\mathcal{A}^m(x)|| \cdot ||\mathcal{A}^n(T^m(x))||.$$

Proof. Let us start from the notation. Denote by π the natural projection from $\mathbb{R}^k$ to the projective space $\mathbb{PR}^k$ and by d the natural metric on $\mathbb{PR}^k$. For a family of convex cones $(C_r)_{r\in J}$, all contained in the interior C^o of another convex cone C, we define their convex hull as

$$\operatorname{conv}(C_r) = \{v \in C; \pi(v) = \pi(\sum_i \alpha_i v_i) \text{ for some } \alpha_i \geq 0, \sum_i \alpha_i = 1, v_i \in C_{r_i}\}$$

The Hausdorff distance in metric d between C and $\operatorname{conv}(C_r)$ equals the supremum of Hausdorff distances between C and C_r (to be absolutely precise, the Hausdorff distance is defined for compact sets and the metric d is defined on the projective space, so we mean here the Hausdorff distance between $\overline{\pi(C)}$ and $\overline{\pi(\operatorname{conv}(C_r))}$). If this supremum is positive (for example, if the cones C_r are continuous as a function of r and J is compact) then this supremum is positive, hence $\operatorname{conv}(C_r) \subset C^o$.

For every $x \in X$ the set $T^{-1}(x)$ is compact. Thus, we can define

$$D_x = \operatorname{conv}(\{C_y; y \in T^{-1}(x)\})$$

for $x \in T(X)$ and, by compactness, we have $D_x \subset C_x^o$. We denote

$$D_x = \operatorname{conv}(\{\mathcal{A}_y(C_y); y \in T^{-1}(x)\})$$

for $x \in T(X)$. We choose D_x as any convex cone contained in C_x^o for $x \in X \setminus T(X)$, we only demand that $x \to D_x$ is a continuous map (this can be done because X is compact, hence $X \setminus T(X)$ is open in X). One can check that, as $D_x \subset C_x^o$, we have

$$\mathcal{A}(x) D_x \subset (\mathcal{A}(x) C_x)^o \subset D_{T(x)}^o.$$

Hence, (D_x) is another invariant cone field, strictly contained in (C_x).

Let for each $x \in X$ d_x be the Hilbert metric in C_x. Let d be the usual metric on $\mathbb{PR}^k$. We have the following properties:

- Each D_x is bounded in d_x. By compactness of X, there exists $K_1 > 0$ such that $\operatorname{diam}_{d_x}(D_x) < K_1$ for all $x \in X$.

- In each D_x the metric d_x is equivalent to d. By compactness of X, there exists $K_2 > 1$ such that for every $x \in X$ for every $v, w \in D_x$ we have $K_2^{-1} d_x(v, w) \le d(v, w) \le K_2 d_x(v, w)$.

- Each $\mathcal{A}(x) : D_x \to D_{T(x)}$ is a contraction. By compactness of X, there exists $\lambda < 1$ such that for every $x \in X$ for every $v, w \in D_x$ we have $d_{T(x)}(\mathcal{A}(x)v, \mathcal{A}(x)w) \le \lambda d_x(v, w)$.

- For $v \in C_x$ denote $\gamma_x(v) = \log(|\mathcal{A}(x)v|/|v|)$. The map $v \to \gamma_x(v)$ is Lipschitz (in metric d) on D_x. By compactness of X, there exists $K_3 > 0$ such that for every $x \in X$ the map γ_x is K_3-Lipschitz (in metric d) on D_x.

- For every $x \in X$ the convex cone D_x contains (for some $v_x \in D_x \cap \mathbb{PR}^k$ and $r_x > 0$) a ball $B(v_x, r_x) = \{w \in \mathbb{PR}^k; d(v_x, w) < r_x\}$. By compactness of X, there exists $r > 0$ such that for every $x \in X$ we have $D_x \supset B(v_x, r)$ for some $v_x \in D_x \cap \mathbb{PR}^k$.

Choose some $x \in X$ and $v, w \in D_x$. Fix $m > 0$. Denote

$$\gamma_x^m(v) = \log \frac{|\mathcal{A}^m(x)v|}{|v|} = \sum_{i=0}^{m-1} \gamma_{T^i(x)}(\mathcal{A}^i(x)v).$$

Note three obvious properties of this function:

- γ_x is a projective function, that is $\gamma_x(v) = \gamma_x(\alpha v)$ for $\alpha > 0$. Thus, we can define γ_x on the projective space $\mathbb{PR}^k$. The same holds for γ_x^m.

- $\gamma_x^m(v) \le \log ||\mathcal{A}^m(x)||$,

- $\gamma_x^{m+n}(v) = \gamma_x^m(v) + \gamma_{T^m(x)}^n(\mathcal{A}^m(x)v)$.

We have

$$\begin{aligned} d(\mathcal{A}^i(x)v, \mathcal{A}^i(x)w) &\le K_2 d_{T^i(x)}(\mathcal{A}^i(x)v, \mathcal{A}^i(x)w) \\ &\le K_2 \lambda^i d_x(v, w) \le K_2 \lambda^i K_1. \end{aligned}$$

Hence,

$$|\gamma^m(v) - \gamma^m(w)| \le K_3 \sum_{i=0}^{m-1} d(\mathcal{A}^i(x)v, \mathcal{A}^i(x)w) \le K_4 := K_1 K_2 K_3 \frac{1}{1-\lambda}$$

for every $v, w \in D_x$.

To finish the proof we need the following lemma.

Lemma 3.5.24. *Let $\mathcal{A} \in GL(k, \mathbb{R})$. Let $K, r > 0$. Assume $|\gamma(v) - \gamma(w)| < K$ for some $v \in \mathbb{PR}^k$ and all $w \in B(v, r)$, where $\gamma(v) = \log |\mathcal{A}v|$. Then there exists a constant $\rho = \rho(K, r)$, depending on K and r but not on A, such that $\gamma(v) \geq \log ||\mathcal{A}|| - \rho(K, r)$.*

Before proving Lemma 3.5.24 let us observe that it indeed implies the assertion of Proposition 3.5.23. As D_x contains some ball $B(v, r)$ with $v \in D_x \cap \mathbb{PR}^k$, we can apply the lemma to the matrix $\mathcal{A}^m(x)$, obtaining $\log ||\mathcal{A}^m(x)|| \leq \rho(K_4, r) + \gamma_x^m(v)$. Hence, for every $w \in D_x$ we have

$$\log ||\mathcal{A}^m(x)|| \leq \rho(K_4, r) + K_4 + \gamma_x^m(w).$$

Similarly, $D_{T^m(x)}$ contains a ball of size r, hence for every $u \in D_{T^m(x)}$ we have

$$\log ||\mathcal{A}^n(T^m(x))|| \leq \rho(K_4, r) + K_4 + \gamma_{T^m(x)}^n(u).$$

Thus, choosing $u = \mathcal{A}^m(x)w$ we get

$$\begin{aligned} \log ||\mathcal{A}^m(x)|| + \log ||\mathcal{A}^n(T^m(x))|| &\leq 2\rho(K, r) + 2K_4 + \gamma_x^{m+n}(w) \\ &\leq 2\rho(K, r) + 2K_4 + \log ||\mathcal{A}^{m+n}(x)| \end{aligned}$$

which is our assertion.

Now, let us come back and prove Lemma 3.5.24.

Proof. We start by a decomposition $\mathcal{A} = O_1 D O_2$, where O_1, O_2 are orthogonal matrices and D is a diagonal matrix with elements $\pm(\sigma_i(\mathcal{A}))$ (the singular values of $\mathcal{A}$). It is enough to prove the assertion for the matrix D.

So, let D be a diagonal matrix. Let e be the eigenvector corresponding to the maximal eigenvalue: $|De| = ||D||$. Even when $v.e = 0$, the ball $B(v, d)$ still must contain a vector w such that $|w.e| \geq 1/2 \cdot \sin r$. We have $w = (w.e)e + (1 - (w.e)^2)^{1/2} e'$, where $e.e' = 0$. Hence,

$$\gamma(w) = \log |Dw| \geq \log(|w.e| \cdot |De|) = \log |w.e| + \log ||D|| \geq \log(\frac{1}{2} \sin r) + \log ||D||.$$

Thus, for every $u \in B(v, d)$ we have

$$\gamma(u) \geq \gamma(w) - K \geq \log ||D|| + \log(\frac{1}{2} \sin r) - K.$$

We are done. $\square$

$\square$

One needs to be careful that quasi-multiplicativity is not equivalent of almost additivity. For instance, let $T : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ be a shift map. We define a linear cocycle $(T, \mathcal{A})$ such that

$$A_0 = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad A_1 = R_\theta,$$

where θ is an irrational angle. It is easy to see that the locally constant cocycle $(T, \mathcal{A})$ is strongly irreducible. Feng [F, Proposition 2.8] showed that the irreducible matrix cocycles are quasi-multiplicative.

We define the *upper joint spectral radius* of $\mathcal{A} : X \to GL(k, \mathbb{R})$ as follows

$$\hat{\rho}(\mathcal{A}) := \lim_{n\to\infty} \sup\{\|\mathcal{A}^n(x)\|^{\frac{1}{n}} : x \in X\}.$$

It is easy to see that $\beta(\mathcal{A}) = \log \hat{\rho}(\mathcal{A})$. Similarly, we define the *lower joint spectral radius* of $\mathcal{A} : X \to GL(k, \mathbb{R})$ as follows

$$\check{\rho}(\mathcal{A}) := \lim_{n\to\infty} \inf\{\|\mathcal{A}^n(x)\|^{\frac{1}{n}} : x \in X\}.$$

We have

$$\log \check{\rho}(\mathcal{A}) = \min\{\alpha_1, \vec{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_k) \in \vec{L}\} = \alpha(\mathcal{A}).$$

Assume that $f : X \to X$ is a convex continuous function on a compact metric space X. We have $\overline{\partial f(\mathbb{R})} = \partial f(\mathbb{R}) \cup \{f'(\infty)\} \cup \{f'(-\infty)\}$, where $\partial f(\mathbb{R})$ is defined as in (2.3.1).

Theorem 3.5.25. *Let (X, T) be a TDS such that the entropy map $\mu \mapsto h_\mu(T)$ is upper semi-continuous and $h_{top}(T) < \infty$. Suppose that $\mathcal{A} : X \to GL(k, \mathbb{R})$ is a matrix cocycle over the TDS (X, T) and $(C_x)_{x\in X}$ is an invariant cone field on X. Then $\alpha(\mathcal{A})$ can be approximated by the Lyapunov exponents of the equilibrium measures for the almost additive potential $t\Phi_{\mathcal{A}}$, where $t \in \mathbb{R}$. Moreover, a minimizing measure for $\mathcal{A}$ exists.*

Proof. Let $\alpha := \alpha(\mathcal{A}) = P'_{\Phi_{\mathcal{A}}}(-\infty)$. We know that $\mathcal{A}$ is almost multiplicative by Proposition 3.5.23.

According to convexity of $P_{\Phi_{\mathcal{A}}}()$, there exists a sequence (t_j) such that $P'_{\Phi_{\mathcal{A}}}(t_j) =: \alpha_j$ exists for every $j \in \mathbb{N}$ and $\alpha_j \to \alpha$ as $j \to \infty$. There exists $\mu_j \in \mathrm{Eq}(\Phi_{\mathcal{A}}, t_j)$ such that $\chi(\mu_j, \Phi) = \alpha_j$ for all j, by Proposition 3.3.7. Let μ be an accumulation [5] point of sequence (μ_j) as $j \to \infty$. By Lemma 3.3.11, we have

$$\chi(\mu_j, \mathcal{A}) \to \chi(\mu, \mathcal{A}) = \alpha.$$

Furthermore, our proof shows that a minimizing measure exists.

□

[5] $\mathrm{Eq}(\mathcal{A}, t)$ is compact in weak-* topology.

Now, we can show the continuity of the minimal Lyapunov exponent.

Theorem 3.5.26. *Let (Σ, T) be a topologically mixing subshift of finite type. Suppose that $\mathcal{A}_n, \mathcal{A} : \Sigma \to GL(k, \mathbb{R})$ are matrix cocycles over (Σ, T), and $\Phi_{\mathcal{A}}$ has bounded distortion. Assume that $(C_x)_{x\in\Sigma}$ is an invariant cone field on Σ. Then, $\alpha(\mathcal{A}_n) \to \alpha(\mathcal{A})$ when $\mathcal{A}_n \to \mathcal{A}$.*

Proof. According to Theorem 3.5.25, $\alpha(\mathcal{A})$ can be approximated by Lyapunov exponents of equilibrium measures for the almost additive potential $t\Phi_{\mathcal{A}}$, where $t \in \mathbb{R}$. Therfore, it is enough to show

$$\chi(\mu_n, \mathcal{A}_n) \to \chi(\mu, \mathcal{A}), \tag{3.5.6.1}$$

where μ, μ_n are the equilibrium measures.

By Proposition 3.5.23, $\mathcal{A}$ is almost multiplicative. Hence, there exist a unique equilibrium measure for the almost additive potential $t\Phi_{\mathcal{A}}$, where $t \in \mathbb{R}$ (see Theorem 2.5.3). Thus, (3.5.6.1) follows from the proof of Lemma 3.5.19. □

Domination can be characterized in terms of existence of invariant cone fields for derivative cocycles (Theorem 2.8.7). This fact shows that 1-domination implies that $\mathcal{A}$ is almost multiplicative. Therefore, one can prove Theorem 3.5.26 for fiber bunched cocycles (see Lemma 2.5.2) under 1-domination assumption.

It is possible to obtain the generalization of Theorem 3.5.25 to the joint spectrum of all Lyapunov exponents. One can also obtain the continuity of the lower joint spectral radius for all Lyapunov exponents.

3.5.7 The proof of Theorem 1.2.6

In this subsection we are going to prove the continuity of the topological pressure for $H_b^r(\Sigma, GL(2, \mathbb{R}))$. In the locally constant cocycles case, Feng and Shmerkin [FS] proved that we have the continuity of the topological pressure. Recently, Park [P] proved that we have the continuity of the topological pressure for typical cocycles. We recall that typical means that a linear cocycle is pinching, twisting and fiber bunching. The techniques we use in the proof are inspired from result [FS]. The result shows that one can prove the continuity of the topological pressure under weaker assumption that Park assumed. The main input our argument is the continuity of Lyapunov expoents that was proved by Backes, Brown, and Butler [BBB] for $H_b^r(\Sigma, GL(2, \mathbb{R}))$.

We use the our result to show that set of $\tilde{\Phi}_{\mathcal{A}}$-equilibrium states for upper triangular matrices that belongs to $H_b^r(\Sigma, GL(2, \mathbb{R}))$ is equal set of equilibrium states its diagonal.

For $s \geq 0$, we define

$$\lambda_e(\mathcal{A}, s) := \lim_{n\to\infty} \frac{1}{n} \int \log \varphi^s(\mathcal{A}^n(x)) d\mu(x),$$

where $\mu \in \mathrm{Eq}(\tilde{\Phi}_{\mathcal{A}}, s)$.

Theorem 3.5.27. *The map* $(\mathcal{A}, s) \mapsto P_{\tilde{\Phi}_{\mathcal{A}}}(s)$ *is continuous on* $[0, \infty) \times H_b^r(\Sigma, GL(2, \mathbb{R}))$.

Proof. Since the topological pressure is upper semi continuous, it is enough to show that it is lower semi-continuous.

Assume that $\mathcal{A}_k \in H_b^r(\Sigma, GL(2, \mathbb{R}))$ and $s_k \in (1, 2)$. We can assume that there is an ergodic measure $\mu \in \mathrm{Eq}(\tilde{\Phi}_{\mathcal{A}}, s)$ by Lemma 3.3.8. Then, by varitional principle

$$\begin{aligned} P_{\tilde{\Phi}_{\mathcal{A}_k}}(s_k) &\geq h_\mu(T) + \lambda_e(\mathcal{A}_k, s_k) \\ &= h_\mu(T) + (2 - s_k)\lambda_e(\mathcal{A}_k, 1) + (s_k - 1)\lambda_e(\mathcal{A}_k, 2). \end{aligned} \tag{3.5.7.1}$$

Notice that

$$\lambda_e(\mathcal{A}_k, 2) = \lim_{n\to\infty} \frac{1}{n} \int \log \| \det(\mathcal{A}_k^n(x)) \| d\mu(x).$$

Therefore, when $\mathcal{A}_k \to \mathcal{A}$, we have

$$\lambda_e(\mathcal{A}_k, 2) \to \lambda_e(\mathcal{A}, 2),$$

and

$$\lambda_e(\mathcal{A}_k, 1) \to \lambda_e(\mathcal{A}, 1).$$

by Theorem 3.4.12. Then, by (3.5.7.1),

$$\liminf_{(\mathcal{A}_k, s_k)\to(\mathcal{A}, s)} P_{\tilde{\Phi}_{\mathcal{A}_k}}(s_k) \geq h_\mu(T) + \lambda_e(\mathcal{A}, s) = P_{\tilde{\Phi}_{\mathcal{A}}}(s).$$

This proves the continuity of $P_{\tilde{\Phi}_{.}}(.)$ at $(\mathcal{A}, s)$. □

Remark 3.5.28. *Recently, C. Freijo and K. Marin [FK2] extended the Backes, Brown, and Butler's result to non-uniformly fiber-bunched cocycles. According to their result, one can prove the above theorem for non-uniformly fiber-bunched cocycles.*

3.5.7.1 Application of Theorem 3.5.27

In the locally constant cocycles case, Falconer and Miao [FM, Theorem 2.5] showed that the set of $\tilde{\Phi}_{\mathcal{A}}$-equilibrium states of upper triangular matrices is precisely the set of $\tilde{\Phi}_{\mathcal{A}}$-equilibrium states its diagonal. Käenmäki and Morris [KM, Proposition 6.2] extended Falconer and Miao's result for higher dimensional case. One can use the Käenmäki and Morris's proof and Theorem 3.5.27 to obtain the following result:

Corollary 3.5.29. *Let $\mathcal{A} \in H^r(\Sigma, GL(2,\mathbb{R}))$ be an upper triangular matrices :*

$$\mathcal{A}(x) := \begin{bmatrix} a(x) & b(x) \\ 0 & c(x) \end{bmatrix}.$$

Then the set of $\tilde{\Phi}_{\mathcal{A}}$-equilibrium states of upper triangular matrices $\mathcal{A}$ is precisely the set of $\tilde{\Phi}_{\mathcal{A}'}$-equilibrium states its diagonal :

$$\mathcal{A}^{'}(x) := \begin{bmatrix} a(x) & 0 \\ 0 & c(x) \end{bmatrix}.$$

Remark 3.5.30. *Recently, Butler and Park [BP] proved some results in this direction for 2-dimensional cocycles.*

Chapter 4

On Hausdorff dimension of thin nonlinear solenoids

4.1 Introduction

In this chapter, we will be concerned with so-called "Smale-Solenoid", a very natural example of a non-conformal map.

Let $M = S^1 \times \mathbb{D}$ be the solid torus, where $\mathbb{D} = \{v \in \mathbb{R}^2 | |v| < 1\}$ carries the product distance $d = d_1 \times d_2$ and suppose $f : M \to M$ such that

$$(x, y, z) \mapsto (\eta(x, y, z) \mod 2\pi, \lambda(x, y, z) + u(x), \nu(x, y, z) + v(x)) \qquad (4.1.1)$$

is a $C^{1+\alpha}$ invective map, where $\lambda(x, 0, 0) = \nu(x, 0, 0) = 0$. Moreover, the component functions η, λ and ν satisfy the following assumption :

1- $\eta^{'}(x, y, z) := \frac{\partial}{\partial x}\eta(x, y, z) > 1$

2- $\lambda^{'}(x, y, z) := \frac{\partial}{\partial y}\lambda(x, y, z) < 1$

3- $\nu^{'}(x, y, z) := \frac{\partial}{\partial z}\nu(x, y, z) < \lambda^{'}(x, y, z),$

at every x, y, z. In addition, the functions λ, ν and $\eta(x, y, z) - d \times x$ are 2π-periodic with respect to x, where d is degree of f. We always assume $d \geq 2$. In the linear solenoid case, if $\eta^{'} < 1/\lambda^{'}$, then such a solenoid can be called a *uniformly thin solenoid*[1].

In fact, our assumption guaranties hyperbolicity. For $\varphi = \eta^{'}, \nu^{'}, \lambda^{'}$, let $\varphi_n(p) = \prod_{i=-n+1}^{i=0} \varphi(f^i(p))$ for $p \in M$ (e.g. $\lambda_n(p) = \prod_{i=-n+1}^{i=0} \lambda^{'}(f^i(p))$).

We will modify assumption of function (4.1.1) in the following subsections.

[1] For the definition of a thick linear solenoid, where $\eta^{'}\lambda^{'} > 1$, see e.g. [Tsu].

Definition 4.1.1. Suppose that W is a differentiable manifold and $X \subset W$. A family $\mathcal{N}$ of smoothly injectively immersed in W manifolds $\{N_\alpha\}_{\alpha \in I}$ (called the *leaves*) is called a *lamination* on X if $N_\alpha \cap N_\beta = \emptyset$ when $\alpha \neq \beta$, $X \subset \cup_\alpha N_\alpha$, and for each $x \in X$ there is a neighborhood U and a homemorphism $h : U \to \mathbb{R}^n$ such that h maps every connected component V of $N_\alpha \cap U$ to $h(V) \cap (\mathbb{R}^k \times \{y\}) \subset \mathbb{R}^n$ for some $y \in \mathbb{R}^{n-k}$.

If $X = W$, this lamination is said to be a *foliation*.

We define hyperbolic attractor for map (4.1.1) as $\Lambda := \cap_{n \in \mathbb{N}} f^n(M)$. $f_{|\Lambda}$ is topologically transitive. We consider the dominated splitting $T_\Lambda(M) = E^u \oplus E^{ws} \oplus E^{ss}$, where ss means strong stable ws weak stable and u stands for unstable.

Definition 4.1.2. For $\varepsilon > 0$ small enough, one defines at each $p \in M$ its *strong stable set*,

$$W^{ss}(p) := \{q \in M, \exists C > 0, \forall n \geq 0, d(f^n(p), f^n(q)) \leq Ce^{-\varepsilon n} Df^n_{|E^{ws}}(p)\}. \quad (4.1.2)$$

In other terms $W^{ss}(p)$ is the set of points whose orbit converge to the orbit of p faster than the contractions $Df^n_{|E^{ws}}$.

We define the natural projection $\pi : p = (x, y, z) \to x(p) := x$. For any set $\mathcal{D} \subset M$, let $p \in \mathcal{D}_x := (\pi_{\mathcal{D}})^{-1}(x)$. Thus, we define stable slice as $\Lambda_x := W^s_{\mathcal{D}_x}(p) \cap \Lambda$.

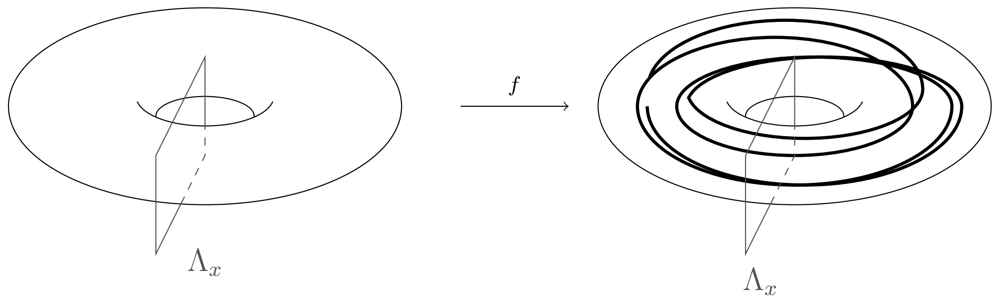


Figure 6: The Solenoid, whose expanding map is a doubling map.

We define $\pi_{(x,y)} := (x, y, z) \mapsto (x, y)$. Suppose that Λ has a *transversal crossing* over the points $(p, q)(q, p \in \Lambda)$. That means,

(1) $\pi_{(x,y)}(p) = \pi_{(x,y)}(q)$,

(2) $\pi_{(x,y)}(W^u_{\text{loc}}(p))$ is transversal to $\pi_{(x,y)}(W^u_{\text{loc}}(q))$.

Sometimes we say that Λ has a transversal crossing over $t = \pi(p) = \pi(q)$.

An *unstable foliation* for Λ is a foliation $\mathcal{W}^u$ of a neighborhood Λ such that

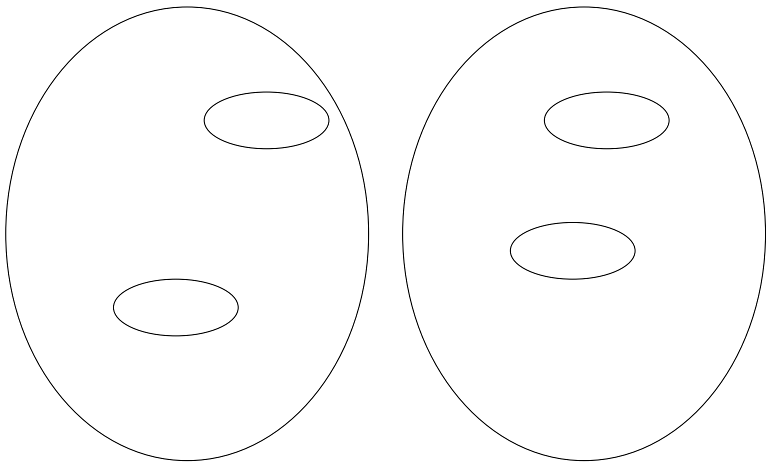


Figure 7: Λ_x.

a) For each $p \in \Lambda$, $\mathcal{W}^u(p)$, the leaf $\mathcal{W}^u$ containing p, is tanget to E^u_p,

b) for each $p \in \Lambda$, $f(\mathcal{W}^u(p)) \supset \mathcal{W}^u(f(p))$.

We say that $\mathcal{W}^u$ is *transitive* (*minimal*), if $\overline{\mathcal{W}^u(x)} = M$ for some (all) $x \in M$.

4.1.1 Unstable Holonomies

Assume that A and B are two nearby embedded disks transverse to unstable lamination W^u then there is a holonomy map defined on a subset of $A \cap \Lambda$ to $B \cap \Lambda$ such that

$$p \mapsto W^u_{\text{loc}}(p) \cap B.$$

In other words, we move along unstable leaves from $A \cap \Lambda$ to $B \cap \Lambda$ (see Figure 8).

These holonomies are always Hölder continuous, but they are not necessary Lipschitz continuous. See for more information [Br].

4.1.1.1 Unstable holonomies for solenoid

The holonomy mapping

$$\Pi^{x(q)}_{x(p)} : W^s_{\mathcal{D}_{x(p)}}(p) \cap \Lambda \to W^s_{\mathcal{D}_{x(q)}}(q) \cap \Lambda \quad (|x(p) - x(q)| < 2\pi)$$

is defined by

$$\Pi^{x(q)}_{x(p)}(p) := W^s_{\mathcal{D}_{x(q)}}(q) \cap W^u_{\text{loc}}(p).$$

Note that there is a unique point the intersection on the right-hand side, by local product structure.

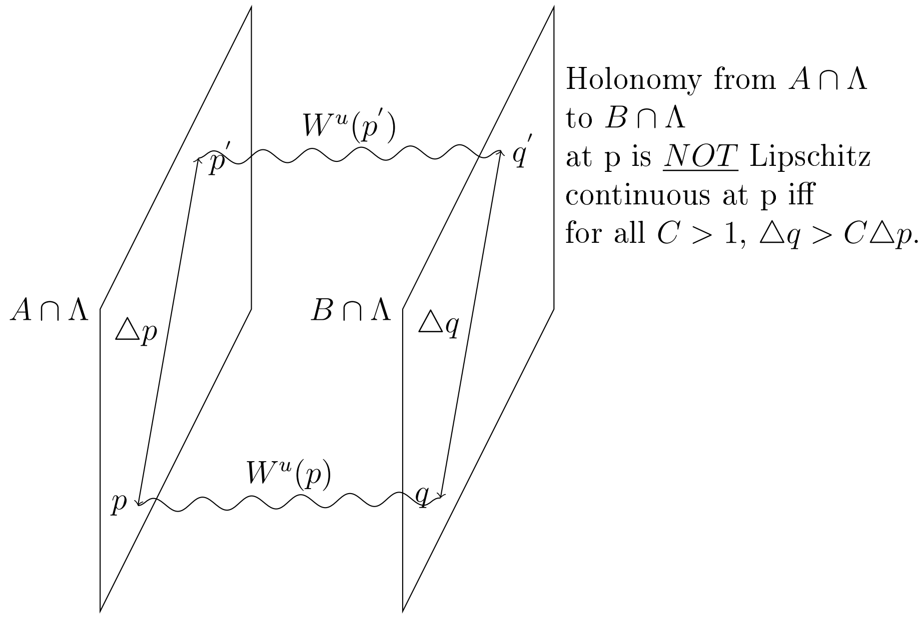


Figure 8: Holonomy

Pinto and Rand [[PR], [PRF]] show that the stable and unstable holonomies of a hyperbolic set Λ of map (4.1.1) have $C^{1+\alpha}$ extension for some α. More precisely,

Theorem 4.1.3 ([PR], [PRF])**.** *Let $f : M \to M$ be $C^{1+\beta}$ with a codimension 1 hyperbolic invariant set Λ which is topologically transitive and has a local product structure. Suppose that the Hausdorff dimension of the unstable leaf segments is one. Then, there is α such that all the holonomies are $C^{1+\alpha}$.*

Due to the integrability of E^s we get stable foliation $\mathcal{W}^s$ of M which is $C^{1+\alpha}$. By Theorem 4.1.3, under an appropriate $C^{1+\alpha}$ change of coordinates becomes the foliation of M by discs x$\times\mathbb{D}$. That means, fixed $p = (x, y, z)$. We consider the map $L : M \to M$ such that

$$p = (x, y, z) \longmapsto (W^s(p) \cap S^1, y, z),$$

which will be used as a suitable change of coordinate.

We obtain a new map $f^{'} = LofoL^{-1}$ such that

$$(x, y, z) \longmapsto (x, h_1(x, y, z) + u(x), h_2(x, y, z) + v(x)),$$

where $h_i^{'}$ s are distinct function for $i = 1, 2$.

The strong stable foliation W^{ss} is known to be $C^{1+\alpha}$ in W^s, see [Br]. Hence, one can find a locally $C^{1+\alpha}$ change of coordinates which would make W^{ss} consist of vertical intervals. In the new coordinates f would be locally $C^{1+\alpha}$ in each W^s, but we do not know what would be its global smoothness.

Keep the assumption of map (4.1.1). According to the above observation, we just assume the map $f : M \to M$ is a $C^{1+\alpha}$ and looks like

$$(x, y, z) \mapsto (\eta(x) \mod 2\pi, \lambda(x, y) + u(x), \nu(x, y, z) + v(x)). \tag{4.1.1.1}$$

Take $w \in M$, the differential of f at $e \in W^s(w)$ is

$$D_e f = \begin{bmatrix} a & 0 & 0 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{4.1.1.2}$$

The mapping f in (4.1.1.1) is called *triangular non-linear.*

The leaves W^u of the lamination $\mathcal{W}^u$ of Λ project locally 1-to-1 to S^1, their angle with all W^s is bounded away from 0. Writing W^u, we consider only a bounded parts of W^u, usually $W^u_{[0,2\pi]}$, which means the part which projects onto $[0, 2\pi]$ 1-to-1 (excepts the ends), in particular 1-to-1 to S^1.

We will always consider f given by the formula (4.1.1.1).

Under transversality and $\chi(\mu_t, \nu') < \chi(\mu_t, \lambda') < -\chi(\mu_t, \eta')$, we show that Hausdorff dimension of the conditional measures on $W^s \cap S^1$ of the geometric equilibrium measure μ_t for f^{-1} and the potential $t_0 \log \lambda'$, where λ' is the weaker contraction rate function, is t_0. Then, we show that the Hausdorff dimension of solenoid attractor is $1 + t_0$.

4.2 Symbolic dynamics and Markov Partitions

It is well known that there is a Markov coding for Anosov diffeomorphisms (see [A]). Markov partitions are a useful way of partitioning the space that a dynamical system acts on by providing a useful tool for developing a "symbolic coding" of $f_{|\Lambda}$. So, we can partition S^1 into closed intervals $I_i = [a_i, a_{i+1})$ where $\{a_i\} = \eta^{-1}(0)$. We write $I_{i_{-n+1},\dots,i_0|} := \bigcap_{j=-n+1}^{0} \eta^{(-j-1)}(I_{i_j})$. Further, we call $V_{|i_1,\dots,i_n} := I_{|i_1\dots i_n} \times \mathbb{D}$ n-vertical cylinder and $H_{i_{-n+1},\dots,i_0|} := f^n(V_{|i_{-n+1},\dots,i_0})$ n-horizontal cylinder, while the set $C_{i_n,\dots,i_0|i_1,\dots,i_n} := H_{i_{-n},\dots,i_0|} \cap V_{|i_1,\dots,i_n}$. We consider $H_{i_{-n+1},\dots,i_0|}(p)$,$V_{|i_1,\dots,i_n}(p)$ and $C_{i_n,\dots,i_0|i_1,\dots,i_n}(p)$, respectively, n-horizontal cylinder, n-vertical cylinder and n-rectangle contain p, respectively. Sometimes, we denote them by $H_n(p), V_n(p)$ and $C^n_{-n}(p)$.

We denote by $H(n)$ and $V(n)$ the sets of all horizontal, respectively vertical, cylinder as above, of generation n. Using projection $\pi_{(x,y)}$, we repeat all the above definitions for coordinate plane (x, y), using the same notation with hat over the symbols. That is,

$$\widehat{H}_{i_{-n},\dots,i_0|} = \pi_{(x,y)} \circ H_{i_{-n},\dots,i_0|}, \ \widehat{V}_{|i_1,\dots,i_n} = \pi_{(x,y)} \circ V_{|i_1,\dots,i_n}, \ ,$$

$$\widehat{C}_{i_n,\ldots,i_0|i_1,\ldots,i_n} = \pi_{(x,y)} \circ C_{i_n,\ldots,i_0|i_1,\ldots,i_n},\ \widehat{W^u} = \pi_{(x,y)} \circ W^u,$$
$$\widehat{W^s} = \pi_{(x,y)} \circ W^s, \widehat{\Lambda} = \pi_{x,y}(\Lambda),\ \ \widehat{f} = \pi_{x,y} \circ f \circ (\pi_{x,y})^{-1}.$$

To construct symbolic dynamics, we introduce symbolic space $\Sigma := \{1, \ldots, k\}^{\mathbb{Z}}$. We consider the canonical coding $\rho : \Sigma \to \Lambda$, where

$$\rho(\underline{i}) = \bigcap_{n=1,2,\ldots} V_{|i_1,\ldots,i_n} \cap \bigcap_{n=1,2,\ldots} H_{i_{-n},\ldots,i_0|}$$

for any two side sequence $\underline{i} = (\ldots, i_{-n}, \ldots, i_0|i_1, \ldots, i_n, \ldots)$.

We denote by Σ^+ and Σ^- the sets of right-sided and left-sided sequence on d symbols which is obtained from the sequence Σ. That is,

$$\Sigma^+ = \{x = (x_i)_i \in \Sigma \mid i > 0\},$$

and,

$$\Sigma^- = \{x = (x_i)_i \in \Sigma \mid i \leq 0\}.$$

If a function $\varphi : \Sigma \to \mathbb{R}$ depends only on the coordinates $\ldots, i_{-n}, \ldots, i_0$, then we will say φ *depends only on the past*. Similarly, if a function $\varphi : \Sigma \to \mathbb{R}$ depends only on the coordinates $i_1, \ldots, i_n, \ldots$, then we will say φ *depends only on the future*.

We also consider the following one side shift maps

$$\sigma^+ : \Sigma^+ \to \Sigma^+ \text{ and } \sigma^- : \Sigma^- \to \Sigma^-.$$

We can use the canonical splitting of our symbolic space $\Sigma = \Sigma^- \times \Sigma^+$, where Σ^- specifies the local unstable manifold and Σ^+ the local stable manifold.

Definition 4.2.1. We say that the unstable foliation is *transverse* if the curves $\pi_{(x,y)}(\rho([\ldots i_{-n} \ldots i_{-1}, i_0|]))$, $\pi_{(x,y)}(\rho([\ldots j_{-n} \ldots j_{-1} j_0|]))$ such that $i_0 \neq j_0$ are transverse. We denote the set of their intersection(crossing) point by Γ. We call these crossing *0order crossing*.

Definition 4.2.2. We call transversal intersection of the curves $\pi_{(x,y)}(\rho([\ldots i_{-n} \ldots i_{-1} i|l_1 l_2 \ldots l_m])$ and $\pi_{(x,y)}(\rho([\ldots j_{-n} \ldots j_{-1} j|l_1 l_2 \ldots l_m])$ for all $1 \leq i, j \leq k, i \neq j$, where

$$\pi_{(x,y)}(\rho([\ldots i_{-n} \ldots i_{-1} i|l_1 l_2 \ldots l_m]) \cap \pi_{(x,y)}(\rho([\ldots j_{-n} \ldots j_{-1} j|l_1 l_2 \ldots l_m]) =$$

$$\pi_{(x,y)}(f^m(\rho([\ldots i_{-n} \ldots i_{-1} i|]))) \cap \pi_{(x,y)}(f^m(\rho([\ldots j_{-n} \ldots j_{-1} j|]))) \cap V_{|l_1 l_2 \ldots l_m} \subset f^m(\Gamma),$$

m-*order crossing*.

Standing Assumption: In throught this chapter, all intersection of this lines (i.e. projections by $\pi_{x,y}$ of unstable manifolds) with different i_0 are transversal.

4.3 On Lipschitz property $\mathcal{W}^u$

In this section we will describe in more details the smoothness of holonomies along unstable leaves.

The discussion here requires only the assumption described at the beginning of previous section (see map 4.1.1.1). It was shown by Anosov [A] for general Anosov systems, the unstable and stable foliations are always $\alpha-$Hölder for some α depending on the rates of expansion and contraction of the system . But, holonomies are not necessary Lipschitz continuous. See for more information [Br].
In the solenoid case, Schmeling [Sch] found solenoids are often lack of regular holonomies but the set of non Lipschitz for unstable foliation are small in the measure sense. We are going to show that it is true in our case.

4.3.1 Strong Lipschitz

Suppose $p \in \Lambda$, $q \in W^u(p)$ are such that the holonomy from $W^s(p)$ to $W^s(q)$ is not Lipschitz at p, i.e., for all $C > 1$ there is a $p^{'} \in W^s_{\mathrm{loc}}(p)$ arbitrary near p such that $\triangle q := d(q, q^{'}) > Cd(p, p^{'}) = C\triangle p$, where $q^{'}$ is the image of $p^{'}$ under the holonomy. We denote by **NL** the set of non-Lipschitz points of the unstable foliation.

Remark 4.3.1. *Notice that by the transversality argument of the sub-bundle E^u all the intersection angles are bounded away from 0, say by α_0. Also by compactness and continuity of E^u on Λ there exists $r_0 > 0$ such that if for $p, p^{'} \in \Lambda \cap W^s$ their mutual Euclidean distance is $r < r_0$ and their i_0 are different then the distance of their $\pi_{(x,y)}$ projections from Γ, more precisely from the intersection $\widehat{W}^u(p) \cap \widehat{W}^u(p^{'})$ which is in particular nonempty, is bounded by $2r/\tan\alpha_0$.*

Compactness and transversality imply that every $p \in$ **NL** has a subsequence of preimages that accumulate exponentially fast to the set of crossing, which is Γ. That means, set of bad points naturally associated with transverse crossing. Hence, we can define set of good points as follows :

Definition 4.3.2. A point $p \in \Lambda$ is said to be *strong locally Lipschitz*, if there is $L > 0$ such that for all n big enough

$$d_\pi(\widehat{f^{-n}}(p), \Gamma \cap \widehat{W}^u_{[-L(\eta_n(p))^{-1}, 2\pi + L(\eta_n(p))^{-1}]}(f^{-n}(p)) \geq L(\eta_n(p))^{-1}, \qquad (4.3.1.1)$$

with the distance in W^u measured between the projections by π to $\mathbb{R}$.

Equivalently we could replace here $\widehat{f^{-n}}(p)$ by $V_n(\widehat{f^{-n}(p)})$. It would influence the constant L only.

By the unstable transversality and transversality of intersection of stable and unstable foliations, this is equivalent to the distance in the $\{(x, y)\}$-plane satisfying

$$d(\widehat{f^{-n}}(p), \widehat{W}^u(p^{'})) \geq \mathrm{Const}(\eta_n(p))^{-1}, \tag{4.3.1.2}$$

for all $p^{'}$ having i_0 different from the i_0 for $f^{-n}(p)$.

We call all points p, which are strong locally Lipschitz with the constant L such that (4.3.1.1) holds for all $q \in W^u(p)$ in place of p, *strong locally bi-Lipschitz.*

Notice that this definition allows to say that the whole $W^u(p)$ is strong locally bi-Lipschitz and write

$$d_\pi(\widehat{f^{-n}}(W^u(p)), \Gamma \cap \widehat{W}^u_{[-L(\eta_n(p))^{-1}, 2\pi+L(\eta_n(p))^{-1}]}(p)) \geq L(\eta_n(p))^{-1}. \tag{4.3.1.3}$$

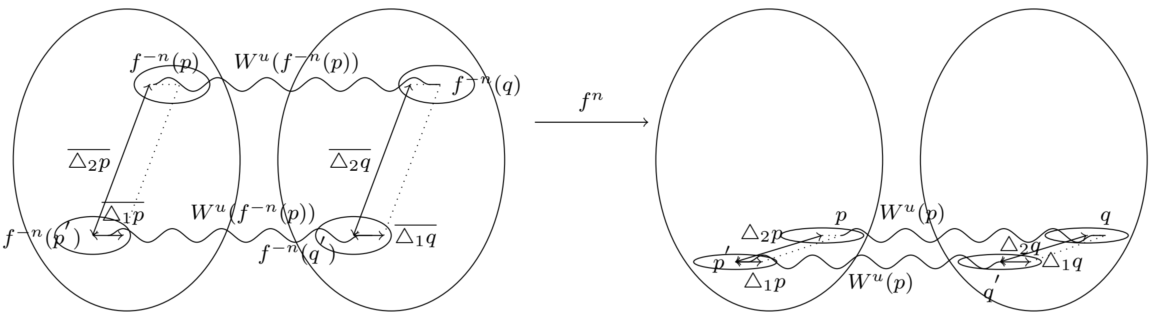


Figure 9: Strong Lipschitz.

We denote the set of all strong locally bi-Lipschitz points in Λ by L^s and $L^s \cap W^s(p)$ with $x(p) = x$ by L^s_x. We call the set complementary to L^s in Λ, *weak non-Lipschitz*, and we denote it by $\mathbf{NL}^{weak}$.

Remark 4.3.3. *Note that if for $\tilde{L} > 0$ strong locally Lipschitz condition $d_\pi(\widehat{f}^{-n}(p), \Gamma \cap \widehat{W}^u(f^{-n}(p))) \geq \tilde{L}(\eta_n(p))^{-1}$ holds and $q \in W^u_{[0,2\pi]}(p)$ then $d_\pi(\widehat{f}^{-n}(p), \Gamma) \geq (\tilde{L} - \mathrm{Const})(\eta_n(q))^{-1}$. Therefore, (4.3.1.1) satisfied at p with $\tilde{L} > 2\,\mathrm{Const}$ strong locally condition holds for all $q \in W^u(p)$, with $L = \frac{\tilde{L}}{2}$. So, p is strong locally bi-Lipschitz.*

Let $d_{pq} := d(\pi(p), \pi(q))$. The following lemma shows that if two point be close to each other, then their trajectories remain close each other.

Lemma 4.3.4 ([HS, Lemma 6])**.** *For $p \in \Lambda$, there is a $C_1 > 0$ such that $\frac{\varphi_n(q)}{\varphi_n(p)} \leq 1 + C_1 d^\alpha_{pq}$ for $q \in W^u(p)$ and $n \in N$ and some $\alpha > 0$.*

Since $\mathbb{D}$ carries the product distance $d = d_1 \times d_2$ we can write $\triangle p = \triangle_1 p + \triangle_2 p$ in a natural way, and likewise $\triangle q = \triangle_1 q + \triangle_2 q$, hence $\overline{\triangle p} := d(f^{-n}(p), f^{-n}(p^{'})) = \overline{\triangle_1} p + \overline{\triangle_2} p$, $\overline{\triangle q} := d(f^{-n}(q), f^{-n}(q^{'})) = \overline{\triangle_1} q + \overline{\triangle_2} q$.

Lemma 4.3.5. *For every $L_2 > 0$ there exists $L_1 > 0$ such that for each p strong locally (bi)Lipschitz with the constant $L = L_1$ there exists $n(p)$ such that for each $q \in W^u_{[0;2\pi]}(p)$ the holonomy between $\Lambda_{x(p)}$ and $\Lambda_{x(q)}$, in $H_{n(p)}(p) \cap \Lambda$, is bi-Lipschitz locally continuous at p with Lipschitz constant L_2, i.e, for every $p' \in \Lambda_{x(p)} \cap H_{n(p)}(p)$ we have*

$$L_2^{-1} d(p, p') \leq d(\Pi^{x(q)}_{x(p)}(p), \Pi^{x(q)}_{x(p)}(p')) \leq L_2 d(p, p'),$$

where d is the euclidean distance in $\mathbb{D}$.

Proof. We repeat (adjust) the calculations in [HS]. We consider $q \in W^u(p)$ and $p' \in W^s(p) \cap \Lambda$. We denote $q' := \Pi^{x(q)}_{x(p)}(p')$. Assume that $p' \in H_n(p) \setminus H_{n+1}(p)$, where $f^{-n}(p), f^{-n}(p')$ are in mutually different H_0's.

Local Lipschitz continuity of the holonomy $\Pi^{x(q)}_{x(p)}(p)$ at p would follow from the existence of a uniform upper bound of

$$\triangle q / \triangle p \tag{4.3.1.4}$$

for p' close enough to p, i.e. n defined above large.

We shall do the estimates in the original coordinates using the triangular form of the differential $Df|_{\{y,z\}} = \begin{bmatrix} \lambda' & 0 \\ a & \nu' \end{bmatrix}$. Due to $\nu' < \lambda'$ we have $Df^n|_{\{y,z\}} = \begin{bmatrix} \lambda_n & 0 \\ a_n & \nu_n \end{bmatrix}$ where $|a_n| \leq \mathrm{Const}\, \lambda_n$. We estimate

$$\begin{aligned} \triangle q =& \bar\lambda_n(q)\overline{\triangle_1 q} + |\bar a_n(q))\overline{\triangle_1 q} + \bar\nu_n(q)\overline{\triangle_2 q}| \\ &\leq \bar\lambda_n(p)\overline{\triangle_1 p} + |\bar a_n(p)\overline{\triangle_1 p} + \bar\nu_n(p)\overline{\triangle_2 p}| + \bar\lambda_n(p)A/\eta_n(p) \end{aligned}$$

for a constant A depending on the angle between W^u and W^s. Here $\bar\lambda_n, \bar a_n$ and $\bar\nu_n$ are averages of derivatives λ_n, a_n and ν_n respectively, on appropriate intervals, namely integrals divided by the lengths of the intervals, horizontal along y for two first integrals and vertical along z for the last one.

On the other hand,

$$\triangle p = \bar\lambda_n(p)\overline{\triangle_1 p} + |\bar a_n(p))\overline{\triangle_1 p} + \bar\nu_n(p)\overline{\triangle_2 p}|.$$

To obtain an upper bound of (4.3.1.4) it is sufficient to assume the existence of an upper bound of the ratio of the above quantities, namely

$$1 + \frac{A(\bar\lambda_n(p)/\eta_n(p))}{\bar\lambda_n(p)\overline{\triangle_1 p} + |\bar a_n(p))\overline{\triangle_1 p} + \bar\nu_n(p)\overline{\triangle_2 p}|}.$$

We needed bars over λ, ν, a to reduce above a fraction to the summand 1. From now on these bars (integrals) are not needed.

We conclude calculations with assuming the existence of an upper bound of

$$\frac{1}{\overline{\triangle_1 p} + (\frac{a_n(p)\overline{\triangle_1 p}+\nu_n(p)\overline{\triangle_2 p}}{\lambda_n(p)})\eta_n(p)} \tag{4.3.1.5}$$

or to assume that the inverse

$$(\overline{\triangle_1 p} + \frac{a_n\overline{\triangle_1 p}+\nu_n(p)\overline{\triangle_2 p}}{\lambda_n(p)})\eta_n(p)$$

is bounded away 0.

Thus, Lipschitz property follows from either of

$$(\frac{a_n\overline{\triangle_1 p}+\nu_n(p)\overline{\triangle_2 p}}{\lambda_n(p)})\eta_n(p) \geq \text{Const} > 0. \tag{4.3.1.6}$$

or

$$\overline{\triangle_1 p}\eta_n(p) \geq \text{Const} > 0 \tag{4.3.1.7}$$

The condition (4.3.1.6), in the diagonal case $a_n = 0$, means that the contraction in the space of stable leaves W^s by f^{-n}, along the coordinate x, due to small $(\eta_n)^{-1}$ is strong enough to bound the twisting effect caused by $\nu_n(p)/\lambda_n(p)$, hence implying the Lipschitz continuity of all the holonomies at p along unstable foliation of a bounded length leaves (e.g. by 2π). This is for $\overline{\triangle_1 p} \approx 0$ (hence $\overline{\triangle_2 p}$ large). Otherwise Lipschitz condition holds automatically.

The condition (4.3.1.7) is equivalent to strong locally Lipschitz (4.3.1.1) in Definition 4.3.2 by transversality condition, see Remark 4.3.1 and (4.3.1.2). This implies that the distance between $W^s(f^{-n}(p))$ and $W^s(f^{-n}(q))$ is bounded by Const$\times\overline{\triangle_1(p)}$ hence $\overline{\triangle_1 q} \leq \text{Const}\overline{\triangle_1 p}$. So, $\triangle q \leq \text{Const}\,\triangle p$, which means Lipschitz property of $\Pi_{x(p)}^{x(q)}$ at p. Notice that Const above large enough we obtain strong bi-Lipschitz property (see Remark 4.3.3).

□

Above lemma implies the following lemma.

Lemma 4.3.6. $\Pi_x^{x'}(L_x^s) = L_{x'}^s$ *for all* $x, x' \in S^1$ *for the holonomy* $\Pi_x^{x'}$ *along unstable foliation. The holonomy is locally Lipschitz on* L^s.

The strong stable set restricts to n-horizontal cylinder is defined by

$$W_{n,\Lambda}^{ss}(p) := (\Lambda \cap W^{ss}(p)) \cap (\cup_{m\geq n} H_m') \cup \{x\},$$

where $H_m' = (H_{i_{-m},\ldots,i_{-1}i_0|} \backslash H_{i_{-(m+1)}i_{-m}\ldots i_{-1}i_0|})$.

Lemma 4.3.7. *There exists* $n \geq 1$ *such that* $W_{n,\Lambda}^{ss}(p) = \{p\}$ *for any* $p \in L^s$.

Proof. Suppose p belongs L^s. That implies, there is a natural number $n(p)$ such that for every $n \geq n(p)$, we have $\pi_{(x,y)}(f^{-n}(p)) \cap \Gamma = \emptyset$. Fix an arbitrary $n \geq n(p)$. Hence, $H_0^{'} \cap W_{0,\Lambda}^{ss}(f^{-n}(p)) = \emptyset$ such that $f^{-n}(p) \notin H_0^{'}$. Therefore, $H_n^{'} \cap W_{n,\Lambda}^{ss}(p) = \emptyset$ due to the f-invarince of Λ and $f^n(W_{0,\Lambda}^{ss}(p)) = W_{n,\Lambda}^{ss}(f^n(p))$ for $p \in \Lambda$. Hence, $W_{n,\Lambda}^{ss}(p) = \{p\}$. $\square$

4.4 Geometric measure

In this subsection we introduce an interesting measure which shows that the set of week Lipschitz is small.

Definition 4.4.1. Let $g : X \to X$ be a continuous on a compact metric space, then two functions $\varphi_1 : X \to \mathbb{R}$ and $\varphi_2 : X \to \mathbb{R}$ are called *cohomologous* on X with respect to g, if there exists a continuous $\zeta : X \to \mathbb{R}$ such that

$$\varphi_1 - \varphi_2 = \zeta - \zeta \circ g \text{ on } X.$$

Sometimes, we use $\varphi_1 \backsim \varphi_2$. Moreover, we say that $\zeta - \zeta \circ g$ is a *coboundary.*

Assume that $\sigma : \Sigma \to \Sigma$ is topologically mixing subshift of finite type. Note that two functions being cohomologous is an equivalence relation. Also observe that if two functions φ_1 and φ_2 are cohomologous then their Birkhoff sums coincide on periodic orbits. Coboundaries are useful since adding a coboundary to a function preserves thermodynamic quantities, as demonstrated by the following result, see in [PU, Chapter 5].

Lemma 4.4.2. *Two Hölder continuous functions φ_1 and φ_2 have the same equilibrium state if and only if $\varphi \backsim \varphi_2 + c$, where $c = P_{\varphi_1} - P_{\varphi_2}$.*

Theorem 4.4.3. *Let φ be a Hölder continuous function on Σ. Then, φ is cohomologous to a function $\widehat{\varphi}$ on Σ^- that depends only on the past. Consequently, they have the same topological pressure and equilibrium measure.*

Let $P(\cdot) = P(f^{-1}, \cdot)$ be topological pressure for the transformation f^{-1} and let $\phi = \log \lambda^{'} \circ f^{-1}$ as potential. We choose $t = t_0$ that is the only zero of the pressure function $t \mapsto P(t \log \lambda^{'} \circ f^{-1})$ (see Figure 10). We call t_0, *affinity dimension* of stable slices.

There is a unique ergodic equilibrium state μ_{t_0} for $t_0\phi$. Moreover, μ_{t_0} has Gibbs properties . We define $h_* := h_{\mu_{t_0}}(f)$.

We replace $\log \lambda^{'} \circ f^{-1}$ by a function having logarithm cohomologous to $\log \lambda^{'} \circ f^{-1}$ (denote it also by $\lambda^{'}$), not depending of future $(|i_1, \dots)$. In conformal case, the quantity t_0 is Hausdorff and box dimensions, here in the non-conformal case is

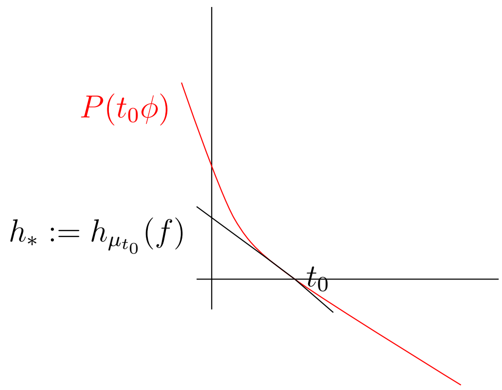


Figure 10: Affinity dimension.

only the upper bound of the dimensions of $W^s \cap \Lambda$, so-called *affinity dimension* [2]. Our main goal is to prove that t_0 is in fact the Hausdorff dimension of all $W^s \cap \Lambda$.

We define $\chi(\mu_{t_0}, \eta^{'}) := \int \log \eta^{'} d\mu_{t_0}$, $\chi(\mu_{t_0}, \lambda^{'}) := \int \log \lambda^{'} d\mu_{t_0}$ and $\chi(\mu_{t_0}, \nu^{'}) := \int \log \nu^{'} d\mu_{t_0}$.

We replace assumption (3) of the map (4.1.1) by $\chi(\mu_{t_0}, \nu^{'}) < \chi(\mu_{t_0}, \lambda^{'})$. In rest of the chapter, we work with above assumption.

4.4.1 General description of measurable partition

We consider (X, τ, γ) probability, complete and separable space. Suppose that $\zeta_1 = \{C_1, \ldots, C_n\}$ is a finite partition of X into sets of positive measure γ, assume that $\tau_1 = B(\zeta_1)$ is the σ-algebra which includes all unions of elements of ζ_1 , so that τ_1 contains 2^n sets. One can get a finer partition ζ_2 and a larger $\sigma-$algebra $\tau_2 = B(\zeta_2)$ whose elements are unions of some, none, or all of the $C_{i,j}$ by partitioning each C_i into $C_{i,1}, \ldots, C_{i,k}$. By iterating this procedure, we have a sequence of partitions

$$\zeta_1 < \zeta_2 < \ldots \tag{4.4.1.1}$$

each of which is a refinement of the previous partition, and a sequence of σ -algebras

$$\tau_1 < \tau_2 < \ldots \tag{4.4.1.2}$$

We consider the limit (4.4.1.1) $\zeta = \vee_{n=1}^{\infty} \zeta_n$. Each element of ζ corresponds to a "funnel"

$$C_{i_1} \supset C_{i_1,i_2} \supset \ldots \tag{4.4.1.3}$$

of decreasing subsets within the sequence of partitions; the intersection of all the sets in such a funnel is an element of ζ.

[2] In fact, t_0 is s in the singular value function (see 2.7).

The sequence (4.4.1.1) is a *basis* because it generates both the $\sigma-$algebra τ and the space X, as follows:

(1) the associated σ-algebras $\tau_n := B(\zeta_n)$ from (4.4.1.2) have the property that $\bigcup_{n\geq 1}\tau_n$ generates τ;

(2) it generates the space X; that is, every "funnel" $C_{i_1} \supset C_{i_1,i_2} \supset \ldots$ as in (4.4.1.3) has intersection containing at most one point (complete property).

Notice that the existence of an increasing sequence of finite or countable partitions satisfying (1) is equivalent to separability of the σ-algebra.

4.4.1.1 Dynamical systems setting

Assume that a map T is an automorphism, i.e., invertible with measure-preserving system. Let ζ be a finite partition of X into measurable sets, and define

$$\zeta_T := \bigvee_{n\in\mathbb{Z}} T^n\zeta = \lim_{n\to\infty}\bigvee_{-n}^{n} T^n\zeta.$$

The elements of this partition are given by $\bigcap_{n\in\mathbb{Z}} T^n C_n$, where $C_n \in \zeta$. Observe that $x \in T^n C_n$ if and only if $T^{-n}(x) \in C_n$, and so knowing which element of $T^n\zeta$ the point x lies in corresponds to knowing in which element of ζ the points $T^{-n}(x)$ lies.

4.4.1.2 Solenoid setting

Fixed $p \in M$. Taking into account to above observation, in the solenoid case (or any general hyperbolic set), we can define for the measure μ a system *stable conditional measure* for the partition into $\Lambda_{x(p)}$ [R].

Let us explain it in more details. Let $W = \{J_1, \ldots, J_n\}$ be a finite partition of S^1 into measurable sets. We consider n-Vertical cylinder $V_{|i_1,\ldots,i_n}$ around the $W^s_{\mathcal{D}}(x)\cap\Lambda$. Given a set $E \subset W^s(p)$, we consider conditional measure $(\mu_{x(p)})_n(E) := \frac{\mu(E\cap V_{|i_1,\ldots,i_n})}{\mu(V_{|i_1,\ldots,i_n})}$. Rokhlin showed $(\mu_{x(p)})_n \to \mu_{x(p)}$ (*weak** topology). They define for μ- a.e. p. We call $(\mu_{x(p)})_W$ *stable conditional measure*. (They of course coincide for $p^{'} \in W^s_{\mathcal{D}}(p)$ when $W^s_{\mathcal{D}}(p) = W^s(p^{'})$).

Similarly, we can define a system conditional measure for strong stable manifold.

4.4.2 Results

Recall that the map $f : M \to M$ is $C^{1+\alpha}$ such that

$$(x, y, z) \mapsto (\eta(x) \mod 2\pi, \lambda(x, y) + u(x), \nu(x, y, z) + v(x)),$$

where $\lambda(x, 0) = \nu(x, 0, 0) = 0$ and

1- $\eta' > 1$

2- $\lambda' < 1$

3- $\chi(\mu_{t_0}, \nu') < \chi(\mu_{t_0}, \lambda')$.

Moreover, the functions λ, ν and $\eta - d \times x$ are 2π-periodic with respect to x, where d is degree of f. We always assume $d \geq 2$.

Now, we can state our main result.

Theorem 4.4.4. *Consider a $C^{1+\alpha}$ map $f : M \to M$ as above, and assume that η' is constant as well as*
1- $\sup \lambda'(p) < (\eta')^{-1}(p) = 1/d$ (d the is degree of η') for $p \in \Lambda$,
2- The unstable lines of the $\pi_{x,y}(\Lambda)$ intersect each other transversal.
Then, $\dim_H(\Lambda) = 1 + \dim_H(\Lambda_x) = 1 + t_0$ for every $x \in S^1$.

Hasselblatt and Schmeling stated in [HS] the following.

Conjecture. *Hausdorff dimension of a hyperbolic set is sum of those its stable and unstable slices.*

In fact, we prove the conjecture for non linear solenoids. Moreover, we can prove Theorem 4.4.4 for much general case.

Theorem 4.4.5. *Assume that η' is not constant. Then Theorem 4.4.4 holds if instead of $\sup \lambda'(p) < 1/d$ we assume $\chi(\mu_{t_0}, \lambda') < \chi(\mu_{t_0}, -\eta')$.*

Definition 4.4.6. A point $p = \rho(\dots i_{-n}, \dots, i_0 | i_1, \dots, i_n, \dots)$ is said to be Birkhoff $(\varphi, \varepsilon, N)$-backward regular for an arbitrary $\varepsilon > 0$ and for $\varphi = \nu', \lambda'$ or η', if for all $n \geq N$

$$e^{n(\chi(\mu.\varphi') - \varepsilon)} \leq \varphi_n(p) \leq e^{n(\chi(\mu.\varphi') + \varepsilon))}. \tag{4.4.2.1}$$

When we mean just (4.4.2.1) we say $(\varphi, \varepsilon, n)$-backward regular, omitting "Birkhoff". Compare Shannon-McMillan-Breiman property in the proof of Lemma 4.5.7.

By bounded distortion the property (4.4.2.1) for $p = \rho(\dots i_{-n}, \dots, i_0 | i_1, \dots, i_n, \dots)$ depends only on $(i_{-n+1}, \dots, i_0)$, provided we insert constant factors before exp, so it can be considered as a property of a horizontal cylinder $H(n)$. Analogously for the forward regularity this is a property of vertical cylinders $V(n)$. We call these cylinders $(\varphi, \varepsilon, n)$-forward or backward regular and all other points or n-th generation cylinders *irregular*.

According to Birkhoff Ergodic Theorem, we can immediately get following corollary :

Corollary 4.4.7. $\mu_{t_0}\{p \mid p \textit{ is backward regular}\} = 1$.

Assume that $f : \Lambda \to \Lambda$ is a $C^{1+\alpha}$ diffeomorphism on the compact locally maximal hyperbolic set and $\varphi : \Lambda \to \mathbb{R}$ is a Hölder continuous function. For each $\alpha \in \mathbb{R}$, we consider the *level set* of Birkhoff averages

$$E_\alpha^-(\varphi) = \{x \in \Lambda, \lim_{n\to-\infty} \frac{1}{n} S_n\varphi(x) = \alpha\},$$

$$E_\alpha^+(\varphi) = \{x \in \Lambda, \lim_{n\to\infty} \frac{1}{n} S_n\varphi(x) = \alpha\}.$$

One can also define the *irregular set* for the Birkhoff averages

$$E_a^b(\varphi) = \{x \in \Lambda, a := \liminf_{n\to\infty} \frac{1}{n} S_n\varphi(x) < b := \limsup_{n\to\infty} \frac{1}{n} S_n\varphi(x)\}.$$

Theorem 4.4.8 ([BV06]). *Suppose that Λ is a compact locally maximal hyperbolic set for $C^{1+\varepsilon}$ diffeomorphism on smooth surface. Then, for each $\alpha \in \mathbb{R}$ and $x^\pm \in E_\alpha^\pm(\varphi)$ we have*

$$\Lambda \cap W^s(x^+) \subset E_\alpha^+(\varphi), \quad \Lambda \cap W^s(x^-) \subset E_\alpha^-(\varphi).$$

And,

$$\dim_H E_\alpha^+(\varphi) = \dim_H(E_\alpha^+(\varphi) \cap W_{\mathrm{loc}}^u(x^+)) + t_s,$$

$$\dim_H E_\alpha^-(\varphi) = \dim_H(E_\alpha^-(\varphi) \cap W_{\mathrm{loc}}^s(x^-)) + t_u,$$

where t_u and t_s are $P(t_s \log df_{|E^s(x)}) = 0, P(t_u \log df_{|E^u(x)}) = 0$.

Theorem 4.4.9. $\dim_H E_a^b(\varphi) = \min_{c\in[a,b]} \dim_H E_c(\varphi)$.

Proof. See [GR, Theorem 1]. □

Thus our main result and above theorems imply that Hausdorff dimension of irregular set is smaller than Hausdorff dimension regular sets.

4.5 $\mathbf{L}^s$ has full support

In this section, we show that Γ is small. Consequently, we will show that set of $\mathbf{NL}^{weak}$ is kind with respect to Hausdorff dimension. We prove the following lemma for the case $\sup\lambda^{'}(p) < 1/d$.

Lemma 4.5.1. $\overline{\dim_B}(\Gamma \cap \widehat{H}_{\underline{i}}) \leq t_0$ *for any* $\underline{i} = (\dots i^{'}_0|)$, *where* t_0 *is affinity dimension of stable slices.*

Proof. Fixed $W^u(\rho(\underline{i}))$ for arbitrary point $\rho(\underline{i})$. We consider n-Horizontal cylinders $H_{i_{-n+1},\dots,i_0|}$ with $i_0 \neq i^{'}_0$ so that $\widehat{H}_{i_{-n+1},\dots,i_0|}$ intersects $\widehat{W}^u(\rho(\underline{i}))$.

We know $P(t_0 \log\lambda^{'}) = 0$. So, by the definition of topological pressure,

$$\sum_n \sum_{\{\widehat{H}_{i_{-n+1},\dots,i_0|}\ \cap\Gamma\neq\emptyset\}} \lambda_n(q)^t \leq C < \infty, \tag{4.5.1}$$

where q is any point such that $\pi_{(x,y)}(q) \in \widehat{W}^u(\rho(\underline{i})) \cap \widehat{H}_{i_{-n+1},\dots,i_0|}$, and for $t > t_0$. Indeed, by transversality, length of intersection the n-Horizontal cylinders $\widehat{H}_{i_{-n+1},\dots,i_0|}$ with Γ is equal diameter n-Horizontal cylinder up to some constant.

We know $\sup_{p\in\Lambda}\lambda^{'}(p) < 1/d$. For each sequence $i := (\dots, i_{-1}, i_0)$ and $r \in (0,1)$, we consider the unique integer $n = n(i)$ such that

$$\mathrm{Length}(\widehat{H}_{i_{-n+1},\dots,i_0|}(q(i)) \cap \widehat{W}^u(\rho(\underline{i})) \leq r \leq \mathrm{Length}(\widehat{H}_{i_{-n+2},\dots,i_0|}(q(i)) \cap \widehat{W}^u(\rho(\underline{i})),$$

[3] where $\pi_{(x,y)}(q(i)) = \widehat{W}^u(\rho(\underline{i})) \cap \widehat{W}^u(\rho(i))$. We can easily verify that for each fixed r the sets

$$I(q(i), n, r) := I_{i_{-n(i)+1},\dots,i_0} = \widehat{W}^u((\rho(\underline{i})) \cap \widehat{H}_{i_{-n(i)+1},\dots,i_0}|.$$

Consider in $\widehat{W}^u(\rho(\underline{i}))$ the ball (arc) $J(q,r) = B(\widehat{q(i)}, r)$. Choose a family $J(q_k, r)$ of the arcs of the form $J(q,r)$ covering $\Gamma \cap \widehat{W}^u(\rho(\underline{i}))$, having multiplicity at most 2, namely that each point in $\widehat{W}^u(\rho(\underline{i}))$ belongs to at most 2 arcs. Then $I(q(i), n, r) \subset J(q_k, r)$ for all k. On the other hand by the definition of $n(i)$ there is a constant K such that $K\,\mathrm{Length} I(q_k(i), n, r) \geq \mathrm{Length}(J(q_k, r))$.

Finally notice that for two different q_k and $q_{k'}$ it may happen that $n = n(k) = n(k^{'})$ and the $n-$th codings $i_{-n+1}, \dots, i_0$ are the same; in other words the $n-$th horizontal cylinders coincide. Then however $J(q_k)$ and $J(q_{k'})$ intersect so the coincidence of these codings may happen only for at most two different k and $k^{'}$.

[3]We mean here the length of the projection by π to $\mathbb{R}$ (of course we can alternatively consider the lengths in $\widehat{W}^u(p)$ or $W^u(p)$).

It follows from (4.5.1) that

$$\begin{aligned}\sum_k (2r)^t \leq & 2K^{-t}\sum_k \mathrm{Length}(I(q_k(i),n,r)) \\ & \leq 2\,\mathrm{Const}\,K^{-1}\sum_{n,k}\lambda_n(q_k)^t \leq \mathrm{Const}\,C < \infty. \qquad (4.5.2)\end{aligned}$$

Hence, as our estimates hold for every $r > 0$, we obtain

$$\overline{\dim_B}(\Gamma \cap \widehat{H}_{\underline{i}}) \leq t_0.$$

$\square$

Definition 4.5.2. For each $\underline{i} = (i_{-n}, \dots i_0|)$ define

$$\begin{aligned} h_n^r(\underline{i}) := & \frac{1}{n+1}\log \#\{(i_1,\dots,i_n) : \\ & \widehat{H}_{\underline{i}} \cap B(\widehat{V}_{|i_1,\dots,i_n}, L_1\eta_n(\pi(\rho(\underline{i})))^{-1} \cap \bigcup_{i'_n,\dots,i'_0 \neq i_0} \widehat{H}_{i'_{-n},\dots,i'_0} \neq \emptyset\},\end{aligned}$$

where L_1 is the constant in Lemma 4.3.5 . Define also

$$h_n^r := \sup h_n^r(\underline{i}), \quad \text{and } h^r := \limsup_{n\to\infty} h_n^r. \qquad (4.5.3)$$

Definition 4.5.3. Let $\underline{i} = (\dots i_0|)$. For $H_{\underline{i}} = W^u(p)$, where $\rho(\underline{i}) = p$, we define

$$\begin{aligned} h_n(\underline{i}) := & \frac{1}{n+1}\log \#\{(i_1,\dots,i_n) : \\ & \widehat{H}_{\underline{i}} \cap \widehat{V}_{|i_1,\dots,i_n} \cap B(\Gamma \cap \widehat{H}_{\underline{i}}, L_1\eta_n^{-1}(\pi(\rho(\underline{i}))) \neq \emptyset\},\end{aligned}$$

Compare (4.3.1.1), and

$$h_n := \sup h_n(\underline{i}), \quad \text{and } h := \limsup_{n\to\infty} h_n. \qquad (4.5.4)$$

Proposition 4.5.4. *h and h^r are independent of L_1 large enough. Moreover, $h \leq h^r$. The opposite inequality holds if* $\sup \lambda' < 1/\sup \eta'$.

We use Lemma 4.5.1 to prove the following lemma.

Lemma 4.5.5. *Keep the assumption Theorem 4.4.4. Then, $h < h_*$.*

[3] I thank Adam Abrams for drawing the picture.

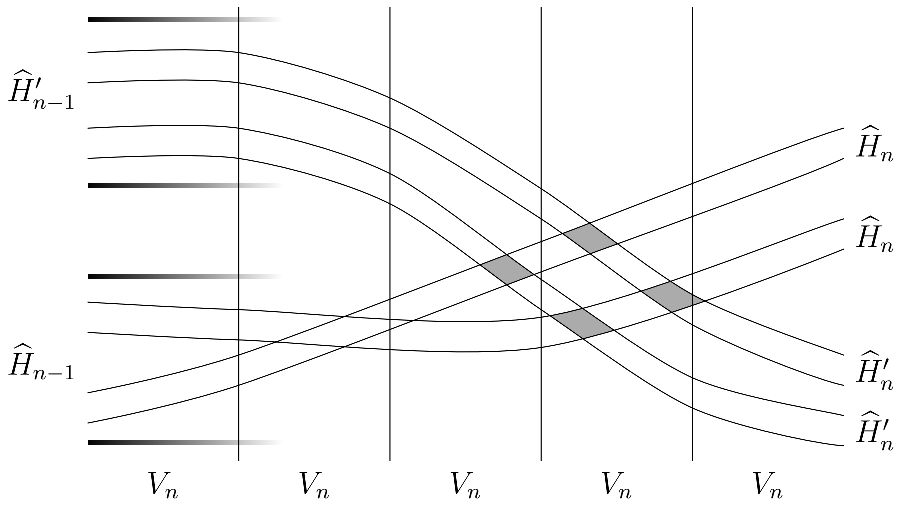


Figure 11: Projection to $(x, y)-$plane. $H_n = H_{i_{-n},\ldots,i_0|}, H_n' = H_{i'_{-n},\ldots,i'_0|}, V_n = V_{|i_1,\ldots,i_n}$.

Proof. For an arability $\varepsilon > 0$ and n large enough, we easily get

$$h_n(\underline{i}) \leq (\overline{\dim}_B(\widehat{H_{\underline{i}}} \cap \Gamma) + \varepsilon))(\log \sup \eta') \tag{4.5.5}$$

for every $\underline{i} = (\ldots i_0|)$.

By Lemma 4.5.1, we have

$$\begin{aligned}\overline{\dim}_B(\widehat{H}_{\underline{i}} \cap \Gamma) \leq & t_0 = \frac{h_*}{-\chi(\mu_{t_0}, \lambda')} \\ & \leq \frac{h_*}{-\sup \log \lambda'} \\ & < \frac{h_*}{\log \sup \eta'}.\end{aligned}$$

Inequality (4.5.2) in Lemma 4.5.1 is uniform, that is n for which it holds is independent $\underline{i}$. So, we can pass $h_n(\underline{i})$ to a uniform version with h_n. Then it completes the proof. □

We used the fact η' is constant in Lemma 4.5.1 to provide concerted scales for $\dim_B$. In general case, the varying scales is obtaining from the partitions of S^1 into arcs between consecutive η^n preimages of a fixed point cause difficulties. They will be overcome by restricting defining h to regular points having $\pi_{x,y}-$images in Γ. We will explain it in next section.

We denote $\mu_{t_0} = \mu$, if there is no confusion.

Remark 4.5.6. *For $p, p^{'} \in L^s$, $(\Pi_{x(p)}^{x(p^{'})})_*(\mu_{x(p)}) = \mu_{x(p^{'})}$.*

We use above fact and $h < h_*$ to prove the following lemma.

Lemma 4.5.7. *If $h < h_*$, then $\mu(\boldsymbol{NL}^{weak}) = 0$. Consequently, $\mu(L^s) = 1$.*

Proof. By applying Shannon-McMillan-Breiman Theorem for f^{-1},

$$\frac{1}{n}\log\mu(H_n(p)) \to h_*$$

for μ almost every $p \in \Lambda$, so for every $\varepsilon > 0$ and n large enough

$$e^{-n(h_*+\varepsilon)} \leq \mu(H_n(p)) \leq e^{-n(h_*-\varepsilon)}. \tag{4.5.6}$$

Given $\varepsilon > 0$ and n, we denote by $Y_{\varepsilon,n}$ the set where (4.5.6) does not hold. Thus, the set

$$Y_\varepsilon^{irr} := \limsup_{n\to\infty} Y_{n,\varepsilon} = \bigcap_n \bigcup_{k\geq n} Y_{\varepsilon,k} \tag{4.5.7}$$

has measure equal 0. Its ε-regular complement $\liminf_{n\to\infty} X_{n,\varepsilon} = \bigcup_n \bigcap_{k\geq n} X_{k,\varepsilon}$ for $X_{k,\varepsilon} = \Lambda \backslash Y_{\varepsilon,k}$ has full measure for each ε.

By Gibbs property measure μ and Birkhoff ergodic theorem for f^{-1} and $\log\lambda^{'}$ in place of Shannon-McMillan-Breiman:

$$\begin{aligned}
\mathrm{Const}^{-1}\, e^{n(t_0+\varepsilon)\chi(\mu,\lambda^{'})} &\leq \mathrm{Const}^{-1}(\lambda_n(p))^{t_0}\\
&\leq \mu(H_n(p))\\
&\leq \mathrm{Const}(\lambda_n(p))^{t_0}\\
&\leq \mathrm{Const}\, e^{n(t_0-\varepsilon)\chi(\mu,\lambda^{'})}.
\end{aligned}$$

The number $e^{(n+1)h^r+nh_*}$ is roughly (that is up to $e^{n\varepsilon}$ order of deviation) an upper bound of the number of horizontal rectangles $H\widehat{(2n+1)}$ whose horizontal extension to $(-L2\pi, (L+1)2\pi)$ intersect $\widehat{f^n(\Gamma)}$ and else which do not belong to $Y_{\varepsilon,n}$.

Indeed, the number $e^{(n+1)h^r}$ comes from $f^n(H)$ for each $H \in H(n+1)$, whereas the number e^{nh_*} comes from the number of regular $H \in H(n+1)$ whose some $H(2n+1) \subset f^n(H)$ belong to $X_{\varepsilon,2n+1} \cap X_{\varepsilon,n}$. Notice that H satisfying this, need not exhaust all H satisfying (4.5.6). The measure μ of each such H is lower bounded for $p \in H$ by

$$\begin{aligned}
\mathrm{Const}\,\lambda_n(p)^{t_0} &= \mathrm{Const}\,\frac{(\lambda_{2n+1}(f^n(p)))^{t_0}}{(\lambda_n(f^n(p))^{t_0}}\\
&\geq e^{n+1((\chi(\mu,\lambda^{'})-3\varepsilon)t_0}\\
&= e^{-(n+1)h_*}e^{-(n+1)3\varepsilon t_0},
\end{aligned}$$

compare (4.6.3), and Gibbs property μ (used already above to reformulate (4.5.6) to the language of λ').

Thus, for the union $\mathcal{H}_{\varepsilon,n}$ these cylinders

$$\begin{aligned}\sum_H \mu(H) &\leq e^{(2n+1)(-h_*+\varepsilon)} e^{(n+1)(h^r+\varepsilon)+n(h_*+3t_0\varepsilon)} \\ &\leq e^{(n+1)(h^l-h_*)+(3n(1+t_0)+2)\varepsilon}\end{aligned}$$

So, for arbitrary $\varepsilon > 0$ small enough,

$$\lim_{N\to\infty} \mu(\bigcup_{n\geq N} (\mathcal{H}_{\varepsilon,n} \cup Y_{\varepsilon,n})) = 0,$$

so $\mu(\mathbf{NL}^{weak}) = 0$.

□

4.6 Proof of the main theorems

Fixed $p \in \Lambda$. We denote the system of conditional measures of γ with respect to partition ζ^s, ζ^{ss}, respectively by $\gamma^s_{x(p)}, \gamma^{ss}_{x(p)}$ and for any measurable set $A \subset \Lambda$ we have

$$\begin{aligned}\gamma^s_{x(p)}(A) &= \gamma^s_{x(p)}(A \cap \zeta^s(p)) \text{ and} \\ \gamma^{ss}_{x(p)}(A) &= \gamma^{ss}_{x(p)}(A \cap \zeta^{ss}(p)).\end{aligned}$$

We denote by $\underline{d}(p)$, $\overline{d}(p)$ the lower and upper pointwise dimensions of γ at p. Since that functions are measurable and $f-$invariant they are constant γ-almost everywhere. We denote these constants by $\underline{d}$ and $\overline{d}$, respectively.

Ledrappier-Young introduced the quantities

$$d^s(p) := \lim_{r\to 0} \frac{\log \gamma^s_{x(p)}(B^s(p,r))}{\log r}, \qquad d^{ss}(p) := \lim_{r\to 0} \frac{\log \gamma^{ss}_{x(p)}(B^{ss}(p,r))}{\log r}, \quad (4.6.1)$$

where $B^s(p,r)$ and $B^{ss}(p,r)$ are balls in stable and strong stable manifolds , provided that corresponding limit exists at $p \in \Lambda$.

Theorem 4.6.1. *For each $x \in S^1$,* $\dim_H(\Lambda_x) = \frac{h_*}{-\chi(\mu_{t_0},\lambda')} = t_0$.

It is convenient to break up the proof of Theorem 4.6.1 into several separate statements. In particular, the theorem is obtained by combining Theorems 4.6.2 and 4.6.3 below.

Theorem 4.6.2. *For each $p \in L^s$,* $\dim_H(\Lambda_{x(p)}) \geq \frac{h_*}{-\chi(\mu_{t_0},\lambda')} = t_0$.

Proof. Suppose that γ is ergodic, hyperbolic measure and $\gamma(L^s) = 1$. Given $p \in L^s$. To estimate the dimension of the set L^s, we study first Hausdorff dimension of γ. We consider $\gamma_{x(p)}$ as the conditional measure of γ.

We may apply the theory of Ledrappier-Young formula [LY2], $h_{\gamma_{x(p)}}(f) = -(d^s - d^{ss})\chi(\gamma, \lambda^{'}) - d^{ss}\chi(\gamma, \nu^{'})$ for γ-a.e. p. According Lemma 4.3.7, since $W^{ss}_{n,\Lambda}(p) = \{p\}$ for $p \in L^s$, we have $d^{ss}(p) = 0$. Therefore, $h_{\gamma_{x(p)}}(f) = -d^s\chi(\gamma, \lambda^{'})$. Moreover, $\dim_H(\gamma_{x(p)}) = \frac{h_\gamma(f)}{-\chi(\gamma,\lambda^{'})}$ by Frostman lemma.

Conclusion, use above arguments for $\gamma = \mu$. Then $\dim_H(\Lambda_{x(p)}) \geq \dim_H(\mu_{x(p)}) = \frac{h_*}{-\chi(\mu_{t_0},\lambda^{'})} = t_0$. □

Now, we prove the other direction.

Theorem 4.6.3. *For each $p \in \Lambda$, $\dim_H(\Lambda_{x(p)}) \leq \frac{h_*}{\chi(\mu,\lambda^{'})} = t_0$.*

Proof. Consider conditional measures $\mu_{x(p)}$ in $\mu-$ a.e. $\Lambda_{x(p)}$ for $p \in \Lambda$. By the definition of $\mu_{x(p)}$ (Gibbs property), we have $\frac{\mu_{x(p)}(V_{|i_1,\dots,i_n} \cap W^s_{\mathcal{D}_{x(p)}}(p))}{\lambda_n^{t_0}(p)} \geq \text{Const}.$ for every $n \in \mathbb{N}$. Consequently, $\dim_H(\Lambda_{x(p)}) \leq \frac{h_*}{-\chi(\mu,\lambda^{'})} = t_0$ for every $x(p)$.

□

Now, we prove Theorem 4.4.4.

Proof. Fixed $(x, y, z) = p \in \Lambda$. We introduce a map F where preserves x coordinate and moves y, x coordinates along holonomy. More precisely, for $q = (x^{'}, y^{'}, z^{'})$,

$$F(x, y, z) = (x, \Pi_x^{x^{'}}(y, z)).$$

We consider $B^s(p, \lambda_n(p))$ for $p \in L^s$ inside stable slice $\Lambda_{x(p)}$. The union images of $B^s(p, \lambda_n(p))$ over all $|x^{'} - x| < 2\pi$ under F becomes Cartesian product. Because the map is locally Lipschitz [4], hence, $\dim_H(\Lambda) \geq 1 + \dim_H(\Lambda_x) = 1 + t_0$.

More precisely F is locally Lipschitz, in the sense that there exists $L > 0$ such that for every $p \in L^s$ there exists measurable $r(p) > 0$ such that for every $r \leq r(p)$ and $q \in B(p, r)$, $\operatorname{dist}(F(p), F(q)) \leq L\operatorname{dist}(p, q)$. This is sufficient to non increase dimension by splitting the space into a countable number of pieces.

Now, we prove other direction. We introduce a measure

$$\widehat{\mu} := \int_x^{x+2\pi} \mu_x d\,\mathrm{Leb}(x), \tag{4.6.2}$$

that its support is in Λ.

[4]By Lemma 4.3.6 all the holonomies $\Pi_x^{x^{'}}$ for $0 \leq x^{'} \leq 2\pi$ are locally bi-Lipschitz on L^s_x.

Given $p \in \Lambda$. We consider image $B^s(p, \lambda_n(p))$ under holonomy, i.e. we move along unstable foliation. Therefore, we have,

$$\{x^{'}\} \times B^s(p, (C+1)\lambda_n(p)) \supset \Pi_x^{x^{'}}(B^s(p, \lambda_n(p))),$$

where $C := \tan \sphericalangle(E^u, E^s)$ [5], and $x^{'} \in B(x(p), \lambda_n(p))$. Hence,

$$\mu(\{x^{'}\} \times B^s(p, (C+1)\lambda_n(p))) \geq \mu(\Pi_x^{x^{'}}(B^s(p, \lambda_n(p))).$$

Then, for any $p \in \Lambda$,

$$\begin{aligned}\widehat{\mu}(B(p, (C+1)\lambda_n(p))) &\geq \int_{-\lambda_n(p)}^{\lambda_n(p)} (\mu_{x^{'}}(H_n(p) \cap \Lambda_{x^{'}}) d\,\mathrm{Leb}(x^{'}) \\ &\geq \mathrm{Const}\,.\lambda_n(p).\lambda_n^{t_0}(p).\end{aligned}$$

So, $\dim_H(\Lambda) \leq 1 + t_0$. □

Proof of Theorem 4.4.5. Now, we explain how to modify the proof of Theorem 4.4.4 such that it works for Theorem 4.4.5.

The technical step involved are roughly as follows. For each n, we consider $H_{n+m}(p)$ Horizontal cylinder for all p's where they are $(\varepsilon, \varphi, n)$ and $(\varepsilon, \varphi, n+m)$ regular points. We come back $n-$step, i.e. consider $f^{-n}(p)$ and we look at all the preimages which is around the intersection. We define entropy for those cylinders contain the points as same as Definition 4.5.3 , and then we show that it is smaller than h_*.

We know that $\varphi_{n+m}(p) = \varphi_n(p)\varphi_m(f^{-n}(p))$. So,

$$\varphi_m(f^{-n}(p)) = \frac{\varphi_{n+m}(p)}{\varphi_n(p)}. \tag{4.6.3}$$

Hence, for p being $(\varphi, \varepsilon, k)-$backward regular for $k = n$, and $k = m+n$, by (4.6.3) and Definition 4.4.2.1, we have

$$\frac{e^{(n+m)(\chi(\mu.\varphi^{'})-\varepsilon)}}{e^{(n(\chi(\mu.\varphi^{'})+\varepsilon)}} \leq \varphi_m(f^{-n}(p)) \leq \frac{e^{(n+m)(\chi(\mu.\varphi^{'})+\varepsilon)}}{e^{(n(\chi(\mu.\varphi^{'})-\varepsilon)}}. \tag{4.6.4}$$

Hence,

$$e^{m(\chi(\mu.\varphi^{'})-\varepsilon(2\frac{n}{m}+1))} \leq \varphi_m(f^{-n}(p)) \leq e^{m(\chi(\mu.\varphi^{'})+\varepsilon(2\frac{n}{m}+1))}. \tag{4.6.5}$$

For each $m, n \in \mathbb{N}$, we denote by $X_{\varepsilon,n,m}$ the union of all $H(n+m)$ Horizontal cylinders of $(\varphi, \varepsilon, n)-$backward regular points in Λ for all $\varphi = \nu^{'}, \lambda^{'}, \eta^{'}$ and yet $(\lambda^{'}, \varepsilon, n+m)$-backward regular points. We call $Y_{\varepsilon,n,m} := \Lambda \backslash X_{\varepsilon,n,m}$ *irregular* points

[5] Angle between stable and unstable manifold is bounded (uniform hyperbolicity)

which $\mu(Y_{\varepsilon,n,m}) \to 0$ as $n \to \infty$. Now, as proof of Lemma 4.5.7, the idea is to remove for each n the irregular set $Y_{\varepsilon,n,m}$ for m to be defined later on, and estimate the number of remaining cylinders $H(n+m)$ which are regular contaminated by other regular cylinders in the sense below (4.6.6).

We say that any point (or cylinder) $p \in H(n+m)$ regular is (Γ^{reg}_{n+m})-*contaminated* if for $\tilde{p} := f^{-n}(p)$

$$\pi_{x,y}(\tilde{p}) \in B^u(\Gamma^{reg}, L_1\eta_n^{-1}(\tilde{p})), \tag{4.6.6}$$

compare Definition 4.3.2. B^u denotes a ball in $\widehat{W}^u(\tilde{p})$. The set Γ^{reg} is defined as Γ in Definition 4.2.1, but restricted to $\widehat{p}$ being $\pi_{x,y}$ image of $q = \rho(\ldots,,i_0|)$ and $q^{'} = \rho(\ldots,,i_0^{'}|)$ such that $f^n(q)$ and $f^n(q^{'})$ are in $X_{\varepsilon,n,m}$.

As in Definition 4.3.2 we can say equivalently that $\widehat{V}_n(\tilde{p})$ is Γ^{reg}_{n+m}-contaminated if it does not satisfy (4.5.3), with Γ replaced by Γ^{reg}_{n+m}.

We are looking for $m > 0$ as small as possible so that

$$\lambda_m(f^{-n}(p)) < (\eta_n(p))^{-1}.$$

Taking into account that both $f^n(q)$ and q are in $X_{\varepsilon,n,m}$. By using (4.6.5),

$$e^{m(\chi(\mu,\lambda^{'})+\varepsilon(2\frac{n}{m}+1))} < e^{n(-\chi(\mu,\eta^{'})-\varepsilon)}.$$

It follows that for $\varepsilon > 0$ small it is sufficient

$$\frac{m}{n} \asymp \frac{\chi(\mu,\eta^{'})}{-\chi(\mu,\lambda^{'})} + \varepsilon^{'} \tag{4.6.7}$$

with $\varepsilon^{'} > 0$ also small.

Now, we do similar to what we did in section 4.5. We briefly explain it. First, for given $H_m(\tilde{p})$ with $p = f^n(\tilde{p}) \in X_{\varepsilon,n.m}$, we define $h_n^{req} := \frac{1}{n+1}\log Z_n$ where Z_n of the number of contaminated vertical cylinder $\widehat{V}(n)$ in $\widehat{H}_m(\tilde{p})$ by dual $\widehat{H}_m(\tilde{q})$ (they are different at zero level).

The number Z_n is bounded by a constant times the number of $H(m)$ above, taking in account L in (4.3.1.1) and the observation that regular H_m, as thinner than $V(m)$ can intersect at most two (neighbor) $V(m)$'s. So, $e^{nh_n^{reg}} \le \text{Const}\,.e^{mh_*}$, hence using (4.6.7),

$$h^{reg} \le h_*(\frac{\chi(\mu,\eta^{'})}{-\chi(\mu,\lambda^{'})}) + \varepsilon^{'}.$$

The argument in the previous theorem now give the analogous statement.

Bibliography

[A] D. V. Anosov, Tangent fields of transversal foliations in U-systems, Math. Notes 2, 1967.

[AEV] A. Avila, A. Eskin and M. Viana, Continuity of Lyapunov exponents of random matrix products, In preparation.

[AV] A. Avila and M. Viana, Simplicity of lyapunov spectra: a sufficient criterion, *Portugaliae Mathematica* **64**: 311–376, 2007.

[BBB] L. Backes, A. Brown, and C. Butler, Continuity of Lyapunov exponents for cocycles with invariant holonomies.*J. Mod. Dyn.,* **12**: 223—260. 2018.

[Bar] L. Barreira, *Thermodynamic formalism and applications to dimension theory*, Progress in Mathematics **294**, Birkhäuser, 2011.

[BP07] L. Barreira and Y. Pesin, *Nonuniform Hyperbolicity: Dynamics of Systems with Nonzero Lyapunov Exponents*, Cambridge University Press, 2007.

[BPS] L. Barreira, Y. Pesin and J. Schmeling, Dimension and product structure of hyperbolic measures, *Annals of Mathematics*, **3**: 755–783, 1999.

[BS] L. Barreira and J. Schmeling, Sets of "non-typical" points havefull topological entropy and full Hausdorff dimension, *Israel J. Math.*, *116*: 29–70, 2000.

[BV06] L. Barreira and C. Valls, Multifractal structure of two-dimensional horseshoes, *Comm. Math. Phys.*, *266*: 455–470, 2006.

[BJKR] B. Bárány, T. Jordan, A. Käenmäki, and M. Rams, Birkhoff and Lyapunov spectra on planar self-affine sets, `https://arxiv.org/abs/1805.08004`.

[Boc] J. Bochi, Discontinuity of the Lyapunov exponents for non-hyperbolic cocycles. `http://www.mat.uc.cl/~jairo.bochi/`.

[Boc1] J. Bochi, Genericity of zero Lyapunov exponents, *Ergodic Theory and Dynamical Systems*, **22**: 1667–1696, 2002.

[Bo] J. Bochi, Ergodic optimization of Birkhoff averages and Lyapunov exponents . *Proceedings of the International Congress of Mathematicians 2018, Rio de Janeiro*, vol. 2, pp. 1821–1842.

[BR] J. Bochi and M. Rams, The entropy of Lyapunov-optimizing measures of some matrix cocycles, *J. Mod. Dyn.*, **10**: 255–286, 2016.

[BG] J. Bochi and E. Garibaldi, Extremal norms for fiber bunched cocycles, *Journal de l'École polytechnique - Mathématiques*, **6**: 947–1004, 2019.

[BGO] J. Bochi and N.Gourmelon, Some characterizations of domination, *Math.Z.*, **263**(1): 221–231, 2009.

[BM] J. Bochi, I.D. Morris, Continuity properties of the lower spectral radius, *Proc. Lond. Math. Soc.*, **110** 477—509. 2015.

[BV] C. Bocker-Neto, and M. Viana. Continuity of Lyapunov exponents for random two-dimensional matrices, *Ergod. Th. & Dynam. Sys.*, **5**, 1413–1442, 2017.

[BGMV] C. Bonatti, X. Gómez-Mont, and M. Viana. Généricité déxposants de lyapunov non-nuls pour des produits déterministes de matrices. *Annales de lInstitut Henri Poincare (C)Non Linear Analysis,* vol. 20, Elsevier, pp. 579–624, 2003.

[Bow] R. Bowen, Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms, Lecture Notes in Mathematics 470, Springer, 1975.

[B] __________, Markov partitions for axiom a diffeomorphisms, *American Journal of Mathematics*, **92**(3): 725–747, 1970

[B1] __________. Topological entropy for noncompact sets, *American Journal of Mathematics*, **184**, 125–136, 1973.

[B2] __________, Some systems with unique equilibrium states, *Mathematical Systems Theory*, **8**(93): 193-202, 1974.

[BV1] C. Bonatti, and M. Viana, Lyapunov exponents with multiplicity 1 for deterministic products of matrices, *Ergod. Th. & Dynam. Sys.*, **24** (5): 1295-1330, 2004.

[Bot] H. Bothe, The dimension of some solenoids, *Ergodic Theory and Dynamical Systems*, **15**: 449–474, 1995.

[Br] A. Brown, Smoothness of stable holonomies inside center-stable manifolds and the C^2 hypothesis in Pugh-Shub and Ledrappier-Young theory, `https://arxiv.org/abs/1608.05886`.

[BS] E. Breuillard and C. Sert. The joint spectrum, `https://arxiv.org/abs/1809.02404`.

[Bu] C. Butler, discontinuity of Lyapunov exponents near fiber bunched cocycles, *Ergodic Theory and Dynamical Systems*, **38**: 523-–539, 2018.

[BP] C. Butler and K. Park, Thermodynamic formalism for $GL_2(\mathbb{R})$-cocycles with canonical holonomies, `https://arxiv.org/abs/1909.11548`.

[BF] H. Busemann and W. Feller. Krümmungsindikatritizen konvexer Flčhen. *Acta Math.*, **66**: 1–47, 1936.

[CFH] Y. Cao, D. Feng, and W. Huang. The thermodynamic formalism for sub-additive potentials. *Discrete Contin. Dyn. Syst.,* **20**(3): 639–657, 2008.

[CPZ] Y. Cao, Yakov Pesin and Y. Zhao, Dimension estimates for non-conformal repellers andcontinuity of sub-additive topological pressure, *Geometric and Functional Analysis*, **29**: 1325—1368, 2019.

[Carath] C. Caratheodory, ,'Uber das lineare Mass von Punktmengeneine Verallgemeinerung des Langenbegriffs', *Nach. Ges. Wiss. Gottingen,* 404–26, 1914.

[CH] J. Chazottes and M. Hochman, On the zero-temperature limit of Gibbs states, *Comm. Math. Phys.*, **297**, no. 1, 265–281, 2010.

[CK] V. Climenhaga and A. Katok, Measure theory through dynamical eyes, `https://arxiv.org/abs/1208.4550`.

[C] V. Climenhaga, The thermodynamic approach to multifractal analysis. *Ergod. Th. & Dynam. Sys.* **34**(5): 1409-–1450, 2014.

[CP] S. Crovisier and R. Potrie, *Introduction to partially hyperbolic dynamics*, Notes, ICTP, 2015.

[CQ] Z. Coelho and A. Quas, Criteria for $\bar{d}$-continuity, *Trans. Amer. Math. Soc. 350,* **8**: 3257–3268, 1998.

[DUZ] A. Davie, M. Urbański and A. Zdunik, Maximizing measures of metrizable non-compact spaces, *Proc. Edinb. Math. Soc. (2).* **50** no. 1, 123–151, 2007.

[DM] L. Douglas and M. Brian, *An introduction to symbolic dynamics and coding*, Cambridge University Press, 1995.

[Fal] K. Falconer, The Hausdorff dimension of self-affine fractals, *Math. Proc. Camb. Phil. Soc.*, **103**: 339–350, 1988.

[F7] K. Falconer, *Techniques in Fractal Geometry,* John Wiley, 1997.

[F8] K. Falconer, Fractal Geometry: Mathematical Foundations and Applications, John Wiley, 2ndEd., 2003.

[FM] K. Falconer and J. Miao, Dimensions of self-affine fractals and multifractals generated by upper-triangular matrices, *Fractals*, **15**(3): 289–299, 2007.

[FFW] A. H. Fan, D. Feng and J. Wu, Recurrence, dimension and entropy, *J. Lond. Math. Soc.*, *64*: 229–244, 2001.

[Feng1] D. Feng, Lyapunov exponents for products of matrices and multifractal analysis. Part I. Positive matrices, *J. Math.*, **138**: 353–376, 2003.

[F] D. Feng, Lyapunov exponents for products of matrices and multifractal analysis. Part II. General matrices, *J. Math.*, **170**: 355–394, 2009.

[F11] F. Feng, Equilibrium states for factor maps between subshifts, *Advances in Mathematics* **226** no. 3, 2470–2502, 2011.

[FH] D. Feng and W. Huang, Lyapunov spectrum of asymptotically sub-additive potentials. *Comm. Math. Phys.,* **297**(1): 1–43, 2010.

[FK] D. Feng and A. Käenmäki, Equilibrium states of the pressure function for products of matrices, *Discrete Contin. Dyn. Syst.*, **30**(3): 699–708, 2011.

[FS] D. Feng and P. Shmerkin, Non-conformal repellers and the continuity of pressure formatrix cocycles, *Geometric and Functional Analysis*, **24**(4): 1101–1128, 2014.

[FK2] C. Freijo and K. Marin, Continuity of Lyapunov exponents for non-uniformly fiber bunched cocycles, `https://arxiv.org/abs/1910.14102`.

[FK1] H. Furstenberg and H. Kesten, Products of random matrices, *the Annals of Mathematical Statistics*, **31**(2): 457–469, 1960.

[GR] K. Gelfert and M. Rams, The Lyapunov spectrum of some parabolic systems, *Erg. Th. and Dyn. Sys.*, **29** 919–940, 2009.

[GU] L. Gurvits, Stability of discrete linear inclusion, *Linear Algebra Appl.*, **231**: 47–85, 1995.

[Hau] F. Hausdorff, Dimension und äußeres Maß, *Math. Ann.*, **79**: 157–179, 1918.

[HS] B. Hasselblatt and J. Schmeling, Dimension product structure of hyperbolic sets, *In Modern dynamical systemsand applications, pages 331–345. Cambridge Univ. Press, Cambridge,* 2004.

[Hut] E. Hutchinson. Fractals and self-similarity, *Indiana Univ. Math. J.*, **30**: 713–747, 1981.

[HL] J. Hiriart-Urruty and C. Lemaréchal, *Fundamentals of convex analysis* ,Springer-Verlag, Berlin, 2001.

[HO] B. Hunt and E. Ott, Optimal periodic orbits of chaotic systems occur atlow period. *Phys. Rev. E54.*, 328—337, 1996.

[IY] G. Iommi, and Y. Yayama, Zero temperature limits of Gibbs states for almost-additive potentials. *J. Stat. Phys.*, **155**: 23–46, 2014.

[J] O. Jenkinson, Ergodic optimization in dynamical systems, *Ergod. Th.& Dynam. Sys.*, **39**(10): 2593–2618, 2019.

[JMU] O. Jenkinson, R.D. Mauldin and M. Urbański. Zero temperature limits of Gibbs-equilibrium states for countable alphabet subshifts of finite type. *J. Stat. Phys.*, **119**: 765–776, 2005.

[KH] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems,* Cambridge University Press, London-New York, 1995.

[KM] A. Käenmäki and I. Morris, Structure of equilibrium states on self-affine sets and strict monotonicity of affinity dimension. *Proc. Lond.Math. Soc.* ,**3** 116 no. 4, 929–956, 2018.

[Ka] B. Kalinin, Livšic theorem for matrix cocycles, *Ann. of Math.*, **173**(2): 1025–1042, 2011.

[KS] B. Kalinin, and V. Sadovskaya, Cocycles with one exponent over partially hyperbolic systems, *Geometriae Dedicata*, **167**(1): 167–18, 2013.

[KS1] B. Kalinin, and V. Sadovskaya, Holonomies and cohomology for cocycles over partially hyperbolic diffeomorphisms, *Discrete and Continuous Dynamical Systems 36*, **1**: 245–259, 2016.

[Ke] T. Kempton, Zero temperature limits of Gibbs equilibrium states for countable markov shifts, *J. Stat. Phys.*, **143**(4): 795-–806, 2011.

[King] J. Kingman, The ergodic theorem of subadditive stochastic processes, *J. Royal Statist. Soc.*, **30**:499—510, 1968.

[1] B. Lemmens, R. Nussbaum, Birkhoff's version of Hilbert's metric and its applications in analysis, `https://arxiv.org/abs/1304.7921`.

[lya] A.M. Lyapunov, *The General Problem of the Stability of Motion, Taylor and Francis Ltd., 1992*, Translated from Edouard Davaux's French translation (1907) of the 1892 Russian original and edited by A.T. Fuller, with an introduction and preface by Fuller, a biography of Lyapunov by V.I. Smirnov, and a bibliography of Lyapunov's works compiled by J.F. Barrett, Lyapunov centenary issue, Reprint of Internat. J. Control 55 (1992), no. 3 [MR1154209 (93e:01035)], With a foreword by Ian Stewart.

[LY1] F. Ledrappier and L.-S. Young, The metric entropy of diffeomorphisms I.Characterization of measures satisfying Pesin's entropy formula, *Ann. of Math.*, *122*(2): 509–539, 1985.

[LY2] F. Ledrappier and L.-S. Young, The metric entropy of diffeomorphisms II.Relations between entropy, exponents and dimension *Ann. of Math.*, *122*(2): 540–574, 1989.

[Mat] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995.

[MMR] P. Mattila, M. Morán, and J. Rey, Dimension of a measure, *Studia Math.*, *142*: 219-–233, 2000.

[M] R. Mohammadpour, Zero temperature limits of equilibrium states for subadditive potentials and approximation of maximal Lyapunov exponent, *Topol. Methods Nonlinear Anal.* (2020). `http://dx.doi.org/10.12775/TMNA.2020.020`.

[Mo] R. Mohammadpour, Lyapunov spectrum properties and continuity of the lower joint spectral radius, `https://arxiv.org/abs/2001.03958`.

[MPR] R. Mohammadpour, F. Przytycki and M. Rams, Hausdorff and packing dimensions and measures for nonlinear transversally non-conformal thin solenoids, `https://arxiv.org/abs/2003.08926`.

[M1] I.D. Morris. Entropy for zero-temperature limits of Gibbs-equilibrium states for countable-alphabet subshifts of finite type, J. Stat. Phys., **126**(2): 315–324, 2007.

[M2] I.D. Morris, A rapidly-converging lower bound for the joint spectral radius via multiplicative ergodic theory, *Adv. Math.*, **225**(6): 3425–3445, 2010.

[M3] I.D. Morris, Mather sets for sequences of matrices and applications to the study of joint spectral radii, *Proc. Lond. Math. Soc.(3)*, **107**(1): 121–150, 2013.

[Ol] E. Olivier, Multifractal analysis in symbolic dynamics and distribution of pointwise dimension for g-measures, *Nonlinearity*, **12**: 1571–1585, 1999.

[Ose] V. I. Oseledets, A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems, *Trans. Moscow Math. Soc.*, **19**:197–231, 1968.

[P] K. Park, Quasi-Multiplcativity of typical cocycles, *Commun. Math. Phys.* (2020). `https://doi.org/10.1007/s00220-020-03701-8`.

[PP] Y. Pesin and B. Pitskel, Topological pressure and the variational principle for noncompact sets, *Fun. Ana. and application.*, **18**(4):307–318, 1984.

[PU1] F. Przytycki and M. Urbański, On the Hausdorff dimension of some fractalsets, *Studia. Math.*, *93*(2): 155-–186, 1989.

[PU] F. Przytycki and M. Urbański, *Conformal fractals: ergodic theory methods,* London Mathematical Society Lecture Note Series, 371, Cambridge University Press, Cambridge, 2010.

[PR] A. Pinto, D. Rand, Smoothness of holonomies for codimension 1 hyperbolic dynamics, *Bull. London Math. Soc.*, 34 : 341–352, 2002.

[PRF] A.A. Pinto, D.A. Rand, and F. Ferreira, dimension bounds for smoothness of holonomies for codimension 1 hyperbolic dynamics, *J. Differ. Equ.*, *243* 168-178, 2007.

[Ragh] M. Raghunathan, A proof of Oseledec's multiplicative ergodic theorem, *Israel J. Math.*, **32**: 356—362, 1979.

[RS] M. Rams and K. Simon, Hausdorff and packing measure for solenoids, *Ergodic theory and dynamical systems*, **23** : 273–292, 2003.

[Roc] R. Rockafellar, *Convex analysis.* Princeton University Press, Princeton, N.J. 1970.

[R] V. Rokhlin, Lectures on the Entropy Theory of Measure-Preserving Transformations, *Russ. Math. Surv. Vol.*, *22*(5): 3–56, 1967.

[RS60] G.-C. Rota and G. Strang, A note on the joint spectral radius, *Indag. Math.*, **22**: 379–381, 1960.

[Ruell] D. Ruelle, Characteristic exponents and invariant manifolds in Hilbert space, *Annals of Math.*, *115*: 243—290, 1982.

[Ru] D. Ruelle, *Thermodynamic formalism: the mathematical structure of equilibrium statistical mechanics*, Cambridge University Press, Cambridge, 2004.

[S] S. Saks, *Theory of the integral.* Instytut Matematyczny Polskiej Akademii Nauk, 1937, **119**: 765–776, 2005.

[Sch] J.Schmeling, Hölder continuity of the holonomy maps for hyperbolic basic sets, *II,Mathematische Nachrichten,170*: 211–225, 1994.

[Si] J. Sinai, Markov partitions and C-diffeomorphisms, *Functional Anal. Appl.*, 2,61–82, 1968.

[Si1] J. Sinai, Gibbs measures in ergodic theory, Russian Mathematical Surveys *27*(4) :21-69, 1972.

[Simon] K. Simon, The Hausdorff dimension of the Smale–Williams solenoid with different contraction coefficients, *Proc. Amer. Math. Soc.*, *125*(4): 1221–1228, 1997.

[So] Solomyak, B. Measure and dimension for some fractal families, *Math. Proc.Camb. Phil. Soc.*, *124*(3): 531–546, 1998.

[Tricot] C. Tricot, Two definitions of fractional dimension, *Math. Proc. Camb. Phil. Soc.*, **91**: 57–74, 1982.

[Tsu] M. Tsujii, Fat solenoidal attractors, *Nonlinearity*, **14**: 1011—1027, 2001.

[V] M. Viana, *Lectures on Lyapunov exponents*, Cambridge University Press, 2014.

[VI] M. Viana, (Dis)continuity of Lyapunov exponents, *Ergodic Theory and Dynamical Systems*, 1–35. doi:10.1017/etds.2018.50.

[Walt] P. Walters, A dynamical proof of the multiplicative ergodic theorem, *Trans. Amer. Math. Soc.*, **335**: 245—257, 1993.

[W1] P. Walters, *An introduction to ergodic theory*, Graduate texts in mathematics. Vol. 79. Springer Science and Business Media, 2000.

[WZ] Q. Wang and Y. Zhao, Variational principle and zero temperature limits of asymptotically (sub)-additive projection pressure, *Front. Math. China.*, **13** (2018), no. 5, 1099—1120.

[Y] J. Yoccoz, Some questions and remarks about $\mathrm{SL}(2,\mathbb{R})$ cocycles, In: *Modern Dynamical Systems and Applications,* Cambridge University Press, Cambridge, 447–458, 2004.

[Z] Y. Zhao, Constrained ergodic optimization for asymptotically additive potentials. *J. Math. Anal. Appl.*, **474**(1):612-–639, 2019.