Referee's report on PhD Thesis of Dr. Jacopo Schino

GROUND STATE, BOUND STATE, AND NORMALIZED SOLUTIONS TO SEMILINEAR MAXWELL AND SCHRÖDINGER EQUATIONS

This PhD thesis deals with semilinear elliptic equations and it is a collection of the papers [2], [3], [4], [5], [6] . More precisely, it focus on two main problems which have recently been in the center of attention of many mathematicians:

1) *curl-curl* problems such as

$$\nabla \times \nabla \mathbf{U} = f(x, \mathbf{U}), \quad \mathbf{U} : \mathbb{R}^3 \to \mathbb{R}^3$$

where $f = \nabla F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is the gradient (with respect to $\mathbf{U}$) of a given nonlinear function $F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$.

2) autonomous Schrödinger systems with constraints of the form

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \partial_j F(u) \\ \displaystyle\int_{\mathbb{R}^N} u_j^2 \, dx = \rho_j^2 \end{cases}$$

for all $j \in \{1, \ldots, K\}$ and $u : \mathbb{R}^N \to \mathbb{R}^K$ with $N, K \geq 1$, $\rho_j \in (0, +\infty)$ are given data, $\lambda_j \in \mathbb{R}$ is part of the unknown being the Lagrange multipliers associated with the L^2 constraint and $F : \mathbb{R}^N \to \mathbb{R}$ is a nonlinear function.

Curl-curl problems find their origins in Maxwell's equations (in 3− dimensional space). However, due to the difficulties to treat these problems, many paper in the last decades consider several simplifications, f.i. by using the so-called *slowly varying envelope approximation*. Nevertheless, such approximations may produce non-physical solutions which do not describe the exact propagation of electromagnetic waves in Maxwell's equations. Other kind of results are obtained with some symmetry assumptions that simplify the problem or with an external potential which belongs into a suitable intersection of Lebesgue spaces.

Solutions to problems like 2) known as *normalized solutions* have been raised much interest in the last decades. The importance of such constraints is due to quantities related to the time dependent Schrödinger systems that are conserved in time (i.e. the energy and the masses). Moreover, the masses have a precise physical meaning since they represent the power supplies in nonlinear optics and the total numbers of atoms in Bose-Einstein condensation. Here the literature is very huge.

Thus, the topics studied here are undoubtedly actual and modern.

The main goal of this work is to generalize the results already known for 1) and 2) by also improving standard techniques.

Indeed, by a suitable control on F, it is proved the existence of a ground state, i.e. a least energy nontrivial solution of problem 1), and the existence of infinitely many geometrically distinct bound states. In particular, multiplicity results for problems like 1) have not been studied so far in $\mathbb{R}^3$. To do so, it is constructed a suitable critical point theory which is also applicable to a wide class of strongly indefinite problems. This

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is a matter of a work in collaboration with J. Mederski and A. Szulkin ([3]) and also published in a prestigious journal.
Then, it is considered also the case $N \geq 3$. Here the role of the symmetry turns out an important tool since it reduces the curl-curl problem to a Schrödinger equation with a singular potential. This is the work based on [2] which is submitted. The most important result contained is about the existence in the case $N = 3$ of a divergent sequence of solutions in the critical case, obtained with the aid of an another group action, which restores compactness. This is indeed the first multiplicity result for curl-curl problems in unbounded domains in the Sobolev-critical case.

In the second part of the thesis problems like 2) are considered. Depending on the assumptions on F the associated energy functional has different behaviours: it is bounded from below for all values of ρ_j (mass-subcritical case), bounded for some values of ρ_j (mass-critical case) and no bounded at all (mass-supercritical case). In this thesis all cases are considered which are object of [4] and [6] both submitted.
In these papers it is proposed a new approach to overcome the problem of the non-compactness of the embedding of $H^1(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ (approach previously introduced for the single equation in the mass-supercritical regime [1]). This approach consists in minimizing the energy functional over a linear combination of the Nehari and Pohozaev constraints intersected with the product of the closed balls in $L^2(\mathbb{R}^N)$ of radii ρ_i, which allows to provide general growth assumptions about F and to know in advance the sign of the corresponding Lagrange multipliers.
At the end of the thesis there is some that joins the two main problems: indeed it is studied normalized solutions for curl-curl problems and to non-autonomous Schrödinger equations with singular potential but with autonomous nonlinearities (argument of [5]).

I conclude that the goals of this thesis are successfully reached and the problems outlined in the introduction were solved. The thesis is written very carefully and in a perfect English, and the proofs are clear and easily understanable. I have found neither mathematical nor formal errors there. The work shows that Dr. Schino is able to perform qualified independent mathematical research and to present the results of his investigation in a cultivated way. Summarizing the above facts, I conclude that the submitted work fulfills the requirements for the PhD Thesis and recommend to issue the degree of PhD to Dr. Schino.

References

BM [1] B. Bieganowski, J. Mederski, *Normalized ground states of the nonlinear Schrödinger equation with at least mass critical growth*, JFA 280 no.11 (2021), 108989.

GMS [2] M. Gaczkowski, J. Mederski, J. Schino, *Multiple solutions to cylindrically symmetric curl-curl problems and related Schrödinger equations with singular potentials*, submitted.

MSS [3] J. Mederski, J. Schino, A. Szulkin, *Multiple solutions to a nonlinear curl-curl problem in* $\mathbb{R}^3$, ARMA (2020), no. 1, 253–288.

MS [4] J. Mederski, J. Schino, *Least energy solutions to a cooperative system of two Schrödinger equations with prescribed* L^2- *bounds: at least* L^2- *critical growth*, submitted.

MS1 [5] J. Mederski, J. Schino, *Normalized solutions to a curl-curl problem with* L^2 *subcritical growth in a bounded domain*, in preparation.

S [6] J. Schino, *Normalized ground states to a cooperative system of Schrödinger equations with generic L^2- subcritical or L^2- critical nonlinearity*, submitted.