Abstract of Filip Rupniewski's PhD thesis "Ranks of tensors, related varieties and rank additivity property for small cases"

The thesis is concerned with ranks of tensors. Topics of central interest are tensor rank, tensor border rank, cactus rank, border cactus rank and related varieties, i.e. secant variety and cactus variety.

In the case of tensor rank and tensor border rank, we analyze the problem of additivity with respect to the direct sum of two independent tensors (of order 3) $p' \in A' \otimes B' \otimes C'$, $p'' \in A'' \otimes B'' \otimes C''$. Namely, we study if the (border) rank of their direct sum is equal to the sum of their individual (border) ranks. For the tensors of order 2 (matrices), tensor rank equals the rank of the corresponding matrix and the additivity holds. In the case of tensors of a bigger order, a positive answer to the problem was previously known as Strassen's conjecture (1973). It was disproved by Shitov (2019). However, his proof was not constructive, and still, an explicit counterexample is not known.

In this thesis, we prove that the additivity of tensor rank holds for some small three-way tensors. For instance, if the tensor p'' is concise and its rank is less or equal dimension of A'' plus 2, then the additivity holds. It is the case also if $p'' \in A'' \otimes (B'' \otimes \mathbb{k}^1 + \mathbb{k}^2 \times C'')$. When we restrict our base field to real or complex numbers, the sufficient condition for rank additivity is that dimensions of both B'' and C'' are equal 3. For $p' \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^3$, $p'' \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^3$ or $p' \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^3$, $p'' \in \mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4$ the additivity also holds. If the base field is $\mathbb{C}$ and the rank of p'' is smaller than 7, it holds as well. As a consequence, the pair of 2×2 matrix multiplication tensors has a rank additivity property. It gives a negative answer to the question of the existence of a faster algorithm for the multiplication of two pairs of 2×2 matrices. The optimal method is to multiply the first pair and then the second one independently.

In addition, we also treat some cases of the additivity of the border rank of tensors. In particular, we show that it holds if the direct sum tensor is contained in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$.

Tensors of a given border rank form a secant variety. Cactus variety is its generalization. It is defined with linear spans of arbitrary finite schemes of bounded length, while secant variety definition uses isolated reduced points only. In particular, any secant variety is always contained in the respective cactus variety, and, except in a few initial cases (when the length is small), the inclusion is strict. It is known that lots of natural criteria on membership in secant varieties are actually only tests for membership in cactus varieties. In this thesis, we propose a pioneering technique for distinguishing actual secant variety from the cactus variety. Our method works in the case of the cactus variety defined for Veronese variety $\nu_d(\mathbb{P}^n)$. We present an algorithm for deciding whether a point in the cactus variety $\kappa_{14}(\nu_d(\mathbb{P}^n))$ belongs to the secant variety $\sigma_{14}(\nu_d(\mathbb{P}^n))$ for $6 \leq d, 6 \leq n$. We obtain similar results for the Grassmann cactus

variety $\kappa_{8,3}(\nu_d(\mathbb{P}^n))$.

For a tensor $p \in \mathbb{C}^k \otimes \mathbb{C}^l \otimes \mathbb{C}^m$ (border) rank of p equals (border) rank of the image of the linear map $(\mathbb{C}^k)^* \to \mathbb{C}^l \otimes \mathbb{C}^m$ induced by p. We extensively use this tool, known as the slice technique, when studying the additivity of (border) rank. We present counterexamples for the slice techniques in the case of cactus rank and border cactus rank. In some sense, the counterexamples which we provide are the smallest possible.

keywords: tensor rank, additivity of tensor rank, Strassen's conjecture, slices of tensor, secant variety, border rank, cactus variety, cactus rank, Hilbert scheme, apolarity

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