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Report on the habilitation project of Jan ROZENDAAL

Jan Rozendaal obtained his PhD in mathematics in 2015, under the supervision of Markus Haase and Ben de Parger in Delft, Netherlands. His thesis was mostly devoted to functional calculus and its applications to semigroups, perturbation theory and approximation theory. Then he was successively assistant professor at the Institute of Mathematics of the Polish Academy of Sciences (IMPAN) for 2 years (2015-2017) and postdoctoral fellow at the Australian National University in Canberra for 3 years (2017-2020). Since august 2020, he is back to an assistant professor position in IMPAN. After his PhD thesis, he worked on three topics: (i) abstract Hardy spaces; (ii) operator valued Fourier multipliers; (iii) stability theory for evolution equations. The main part of Rozendaal's habilitation essay describes his work on the topics (ii) and (iii). The latter is based on a long-term collaboration with Mark Veraar, from Delft and also on a collaboration with David Seifert, from Oxford, and Reinhard Stahn, from Dresden.

I will now report on the scientific achievements of Jan Rozendaal regarding the 5 papers (R1)–(R5) from his application. I first want to say that the habilitation essay is very well written and explains in a bright way the connections betweeen the different papers, the motivations for the development of certain topics, as well as the state of the art.

Let X, Y be Banach spaces and let $L(X,Y)$ denote the Banach space of all bounded operators from X into Y. Let $n \geq 1$ be an integer and let $1 \leq p, q < \infty$ be two indices. Let $L^p(\mathbb{R}^n; X)$ denote the Bochner space of p-integrable X-valued functions on $\mathbb{R}^n$. Let $\mathcal{F}(f)$ denote the Fourier transform of a function f defined on $\mathbb{R}^n$, whenever it exists. By definition, a function $m\colon \mathbb{R}^n \to L(X,Y)$ induces a bounded $(L^p(\mathbb{R}^n; X), L^q(\mathbb{R}^n; Y))$-Fourier multiplier provided that there exists a bounded operator $T_m\colon L^p(\mathbb{R}^n; X) \to L^q(\mathbb{R}^n; Y)$ such that

$$T_m(f) = \mathcal{F}^{-1}\big(m \cdot \mathcal{F}(f)\big)$$

for any f belonging to a dense subspace of $L^p(\mathbb{R}^n; X)$. The study of Fourier multipliers in the scalar case ($X = Y = \mathbb{C}$) for $p = q$ is a old and rich topic which forms an important part of Harmonic Analysis on Euclidean spaces. Among the most famous contributions, one can mention Marcinkiewicz's Theorem and Mikhlin's Theorem from the 50's. During the 80's, pioneering work of Burkholder and Bourgain led to various generalizations of Mikhlin's Theorem to the vector valued case under the following restrictions: X, Y are UMD spaces and m is scalar-valued. In 2001, Weis extended Mikhlin's Theorem to the case when X, Y are UMD spaces and m is allowed to be operator valued. A key assumption in Weis's Theorem in an R-boundedness property which turns out to be an avoidable ingredient. Weis's Theorem was the start of an intense period of activity on $(L^p(\mathbb{R}^n;X), L^p(\mathbb{R}^n;Y))$-Fourier multipliers (here $q=p$), and also on R-boundedness.

The case when $q \neq p$ is interesting only when $p < q$. The paper (R4) is the first one to establish non trivial results on $(L^p(\mathbb{R}^n;X), L^q(\mathbb{R}^n;Y))$-Fourier multipliers for $p < q$. The main result of this paper is the following: *Assume that X has type $p_0 \in (1,2]$ and Y has cotype $q_0 \in [2,\infty)$. Let $p \in (1,p_0)$, $q \in (q_0,\infty)$ and $r \geq 1$ such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. If $m\colon \mathbb{R}^n \to L(X,Y)$ is such that $\{|t|^{\frac{n}{r}} m(t) \, : \, t \in \mathbb{R}^n \setminus \{0\}\}$ is R-bounded, then m induces a bounded $(L^p(\mathbb{R}^n;X), L^q(\mathbb{R}^n;Y))$-Fourier multiplier.* The statement and the proof of this theorem emphasize the role of R-boundedness, although the approach is completely different from Weis's one (note that there is no UMD assumption on either X or Y). The so-called γ-spaces, introduced by Kalton and Weis at the beginning of the 2000's, play a major role. The above theorem provides a new interaction between Fourier multipliers and the type/cotype properties. The role of type/cotype for the study of γ-spaces had been recognized by Veraar in 2011 and is remarkably exploited here. The paper (R4) also contains Fourier multiplier results in which the type/cotype asssumptions are replaced by Fourier type assumptions, as well as a specific study of $(L^p(\mathbb{R}^n;X), L^q(\mathbb{R}^n;Y))$-Fourier multipliers in the case where X and Y are Banach lattices. Altogether, (R4) is a very innovative paper.

A remarkable paper by Amann published in 1997 initiated the study of operator valued Fourier multipliers between vector valued Besov spaces. By now it is well-known that multipliers theorems on Besov spaces $B^s_{p,v}(\mathbb{R}^n;X)$ have applications to boundary-value problems, maximal regularity, stability theory and functional calculus theory. It was therefore natural to investigate analogs of the results of (R4) when the scale of L^p-spaces is replaced by the scale of Besov spaces. This is achieved in the paper (R5). The latter contains several sharp results, either in the Fourier type setting or in the type/cotype setting. The paper exploits in a clever way the case of L^p-L^q multipliers for functions with compact support.

Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on some Banach space X and let $-A$ denote its infinitesimal generator. For $x \in X$, $t \mapsto T_t(x)$ is the solution of the abstract Cauchy problem associated with A, with initial value x. This is why it is crucial to be able to describe the asymptotic behaviour of $T_t(x)$ in specific cases. This behaviour may depend on x, more precisely on the membership of x to certain subspaces of X. The two papers (R3) and (R2) are mostly devoted to this topic, and various applications. A key connection relating this topic to (R1) is given by the fact that certain asymptotic properties $T_t(x)$ are implied by the boundedness of $(L^p(\mathbb{R}^n;X), L^q(\mathbb{R}^n;Y))$-Fourier multipliers associated with functions $m\colon \mathbb{R}^n \to L(X,Y)$ defined in terms of the resolvent of A. The papers (R3) and (R2) therefore emphasize the role of decay properties of the resolvent of A to get asymptotic properties of $T_t(x)$. One of the main results of (R3) is the following, where $R(\lambda, A) = (\lambda - A)^{-1}$ denotes the resolvent of A: *Assume that X is a Hilbert space and that the spectrum of A is included in $\mathbb{C}_+$. Assume further that $\|R(\lambda,A)\| \leq C(1+|\lambda|)^\beta$ for some $\beta > 0, C \geq 0$ and all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \leq 0$. Then for each $\tau \geq \beta$, there exists $C_\tau \geq 0$ such that $\|T_t A^{-\tau}\| \leq C_\tau t^{1-\frac{\tau}{\beta}}$ for all $t \geq 1$.* Such a result provides an estimate of $\|T_t(x)\|$ when $t \to \infty$, whenever x belongs to the domain of the fractional power $A^{-\tau}$. The paper (R3) also contains versions of this result under Fourier type or under type/cotype assumptions on X, which fully exploit (R1). The paper (R3) also includes examples illustrating the novelty of the results obtained.

Following a similar line of attack, the paper (R2) shows that for $(T_t)_{t\geq 0}$ and A as above, if X is a Hilbert space, then the growth rate of the resolvent of A near the imaginary axis implies a corrresponding growth rate of T_t. More precisely: *If* $g\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$ *is a non-decreasing function such that* $\|R(\lambda, A)\| \leq g\big(\mathrm{Re}(-\lambda)^{-1}\big)$ *for* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda) < 0$, *then there exists a constant* $C > 0$ *such that* $\|T_t\| \leq Cg(t)$ *for all* $t \geq 1$. Further this theorem is extended to the case when X is an arbitrary Banach space and $(T_t)_{t\geq 0}$ is asymptotically analytic. Another version of the theorem is given in the case when X in an L^p-space and $(T_t)_{t\geq 0}$ is a positive semigroup. This leads to interesting applications, in particular to a wave equation.

The problem considered in the paper (R5) is close but different from the one discussed in the previous paragraph. Let $(T_t)_{t\geq 0}$ be a bounded C_0-semigroup on some Hilbert space X, with generator $-A$. Assume that $i\mathbb{R}$ is included in the resolvent set of A and consider $M(t) = \sup\big\{\|R(is, A)\| \,:\, s \in \mathbb{R},\ |s| \leq t\big\}$ for $t \in \mathbb{R}_+^*$, which measures the behaviour of the resolvent on A on the imaginary axis. Assume that $M(t) \to \infty$ when $t \to \infty$. A remarkable result of Batty-Duyckaerts (2008) asserts that there exist $c, C > 0$ such that

$$\frac{c}{M^{-1}(Ct)} \leq \|T_t A^{-1}\| \leq \frac{C}{M_{\log}^{-1}(ct)}$$

for large t. Here $M_{\log}$ is a modification of M by a logarithmic factor. This double estimate describes the behaviour of $\|T_t(x)\|$ for x in the domain of A. It led to the question whether the right handside $\frac{C}{M_{\log}^{-1}(ct)}$ could be replaced by $\frac{C}{M^{-1}(ct)}$ in order to have a sharp estimate of $\|T_t A^{-1}\|$. This problem was solved for special (but important) functions M by Borichev-Tomilov in 2010. The paper (R5) goes much further and provides a complete solution. The main result is the following: *(i) If M has "positive increase", then there exist $c, C > 0$ such that*

$$\frac{c}{M^{-1}(Ct)} \leq \|T_t A^{-1}\| \leq \frac{C}{M^{-1}(ct)}$$

for large t. (ii) For normal semigroups, the positive increase assumption on M is necessary to have the above estimate. The positive increase assumption is an estimate $M(ts) \geq Kt^{\alpha}M(s)$ for some $\alpha > 0$ and sufficiently large $t, s > 0$. The above double result is an outstanding achievement, which already had a strong impact in the semigroup community. The paper (R5) also considers the cases where the resolvent of A has a singularity at 0, and at 0 and ∞ at the same time; it contains sharp results of the same flavor.

Jan Rozendaal wrote and published 13 papers, including (R1)–(R5), in international refereed journals, some of them in top journals such as Advances in Mathematics, Transactions of the American Mathematical Society, Journal of Functional Analysis (2 times) and Revista Matematica Iberoamericana. He is a specialist of functional calculus for either sectorial operators or half-plane type operators. He has several striking results on this topic, not mentioned in the above scientific description. Furthermore, his two recent papers on Hardy spaces open up some interesting new prospects. They also show his ability to explore new topics and to broaden his fields of expertise.

Jan Rozendaal was invited in many international conferences and took part to the organization of one of them. He was also involved in the scientific life of his department at the Australian National University. He has a strong experience in teaching but I do not have any relevant information regarding this activity.

In conclusion, I have an excellent option on Rozendaal's achievements. He is working on important topics, he has several first class results and collaborates with the best specialists of his domain. The development of his scientific carreer is extremelly positive. He has been able to carry out various projects in a very coherent approach and to prove significant results on each of them. He is an independent researcher, able to direct a thesis. I strongly recommend awarding the habilitation to Jan Rozendaal.

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