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REPORT

on PhD thesis of **Reza Mohammadpour Bejargafsheh** entitled

"Thermodynamic formalism and multifractal analysis for matrix cocycles and solenoids"

PhD thesis of Reza Mohammadpour consists of 84 pages and is divided into 4 main sections. It contains bibliography, abstract (in Polish and English), acknowledgments and keywords. Thesis was prepared under supervision of Michał Rams (IMPAN). The thesis is written in English. Core part of the thesis are 3 articles (co)authored by Reza Mohammadpour:

[1] R. Mohammadpour, *Zero temperature limits of equilibrium states for subadditive potentials and approximation of the maximal Lyapunov exponent*, Topol. Methods Nonlinear Anal., **55(2)** (2020), 697-710.

[2] R. Mohammadpour, *Lyapunov spectrum properties and continuity of the lower joint spectral radius*, arXiv:2001.03958

[3] R. Mohammadpour, F. Przytycki, M. Rams, *Hausdorff and packing dimensions and measures for nonlinear transversally non-conformal thin solenoids*, arXiv:2003.08926

Considered topics are bond together by such notions as entropy, pressure or Hausdorff dimension, all analyzed from ergodic theory point of view. Topics come from active field of research, with various nontrivial tools, methods and constructions. The author tries to give a wide overview on what was done before, what is the motivation, how his generalizations of techniques follow from the literature or differ from standard approach which does not work. There is a large section of Preliminaries which collects huge number of useful facts and notions. Then section 3 deals with Lyapunov exponents, topological entropy and pressure for subadditive potentials and cocycles. The last section deals with "solenoidal" attractors embedded in solid torus. All the results have references to recent works published in very good journals by prominent mathematicians. The presented research is timely and relevant for the field of dynamical systems.

Unfortunately, this very positive first impressions completely disappears after more careful read of the content of the thesis. When reading the thesis I had an impression that the author by mistake presented a preliminary version of the manuscript. It is a kind of compilation of content of articles, all using different notation, together with many typesetting problems. Many crucial arguments are not explained, which can be accepted in research article in top journal but not in a PhD thesis, which in my opinion should be as much self consistent as possible. In my opinion the thesis in present form does not satisfy formal nor traditional requirements for PhD theses. It should be carefully revised and resubmitted for evaluation. On the other hand, I am sure that the author is able to make such revision.

Below, let me present (by no means complete) list of problems I found when reading the thesis.

1^{10} What are "..." in last bullet point standing for? Is the list complete?

2^{3} In this context, it should be "endowed with" not "by".

2 In the definition of Birkhoff average, add "provided it exists"

2_{12} $T^n)(x)$ should be $T^n(x))$

2_6 Is "does not exist for all points" a correct statement? Or maybe it does not exists for some points?

2_2 It is better to write "such that" in place of "and". I also suggest to enumerate this formula, since L is used several times in much further parts of the thesis. So it is easy to overlook this definition. It is worth to recall it e.g. in 1.2.1.

3^3 Since Z is not compact, it is necessary to state which definition of h_{top} is going to be used (there are several non-equivalent approaches in the literature).

3^6 The statement "It can also be characterized as" is not completely trivial. Some comment should appear here.

3^{12} Equilibrium measure is not defined.

4^3 What does it mean

$$E_f(\alpha) \neq \emptyset \Leftrightarrow \Omega := \{....\}?$$

Is the condition about entropy below? Do you mean ".... $\Leftrightarrow$... where $\Omega := ...$"? What is the exact meaning of $\forall \alpha \in \Omega$? Since α appears on both sides of $\Leftrightarrow$ it is not completely clear. Does it mean that equivalence is true for every α in Ω or both sides are true for all α simultaneously.

4^{10} Sentence "More precisely, given..." ends suddenly without any conclusion.

4^{13} $(u, v) \mapsto X \times \mathbb{R}^k$ is without sense

4_2 what does it mean $\log^+ ||A|| \in L^1(\mu)$? Is $\log^+$ the same as log later in the thesis, e.g. in (1.2.4)?

6^1 "such that" or "by"?

6^6 What is $\mathcal{L}$? What are I, J, K and in particular, what is $A(I)$? what is pinching and twisting a few lines later?

10^8 "everywhere".

12_3 When defining W^s_{loc} you fix (without giving it explicitly) that full shift is endowed with a specific metric. Please write it down. It is quite important in some places, e.g. for ω

13_7 Is $\mathcal{L}_n$ the same as $\mathcal{L}(n)$ defined on previous page? There are various places where the thesis is lacking consistency in notation.

13_3 "For systems like hyperbolic systems which there is a Markov coding for them" does not look as proper English sentence. Was Markov coding defined yet? Or hyperbolic system?

15^4 Usual way of putting this type of citation is [S, Theorem 7.9]. Latex formats it in an automatic way.

15_4 Symbol L is already used for an important notion, so should be avoided in other contexts.

16_3 I personally prefer statement of Multiplicative Ergodic Theorem as in Walters book. Does "flag" has any special meaning? Does $>$ mean $\supset$? How to understand that $x \mapsto V^i_x$ is measurable? Don't we need the fact that Lyapunov exponents are "nicely" ordered?

16_{12} What does it mean $||\det \mathcal{A}^n(x)||$?

19^{12} Is there any good reason to write $P(q\psi)$? Two lines later, and probably everywhere else, it becomes $P(t\psi)$

19_6 What is P_n in the definition? How is the equality $= \sup ...$ in the line below obtained? Maybe it is the missing definition?

20^4 Was $\chi(\mu, \Phi)$ defined somewhere?

27^5 I cannot find definitions of $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$. Additionally, replace s by u in the second symbol.

32_5 "... is represented by $\binom{k}{l} \times \binom{k}{l}$" - what is this symbol? Was it defined?

33^1 "We recall..." - was it defined before, so that now you recall the definition? What norm is used there?

33^3 "... such that" should be "by"

33_6 "We remark that supremum (3.2.1) is attained..." - is it obvious? Or maybe a theorem?

34^9 What is $\vec{\alpha}$? Is it somehow related to α from 5 lines above? Let me also mention that α was used before to denote partitions, e.g. on page 10, and is somehow "reserved symbol" in definitions related to Birkhoff spectrum (e.g. see line 5 on page 3).

35^2 Why x not $\vec{x}$? Is there any particular rule when we use arrow or not? The same in 45^8 and further.

35^{15} "Assume that .... is subadditive potential ...." - should not some sum appear in $\{q_i \log \phi_{n,i}\}$? If not, why do we have sum in the first formula of $\vec{q}.\vec{\Phi}$? By the way, aren't we defining a notion used several times later on? Maybe it is worth emphasizing?

35_9 Symbol $\beta(\Phi)$ was newer defined. I guess from the context, it is not sequence of $\beta(\phi_n)$ where $\beta(f)$ is from page 3.

37_{13} "Let $\mu \in M(X,T)$..." suggest that we fix μ. It is not the case here.

37_3 What do we exactly "show" here?

37_1 For consistency, symbol $\mathcal{A}$ should be used instead of A. Also note that $||A-B||$ may have different meaning than $||A(x)-B(x)||$. What is $h_r(A-B)$ in the definition of D_r? $A-B$ is not necessarily invertible.

38_{14} What is hyperbolicity constant?

38_6 How is $D\mathcal{A}_v$ exactly defined? What is "the action $\mathbb{PR}^k \to \mathbb{PR}^k$"?

39^3 What is $W^{s/u}_{loc}$? Is the same y used in both formulas below?

39^6 It would be useful to provide exact reference, like [KS, Theorem ....] not only [KS].

39^{10} It is worth to extend the formula on $H^s_{x\leftarrow y}$ using $A^{-1}_{x_0}$ in place of $\mathcal{A}(x)^{-1}$ etc. Then some statements became more apparent. By the way, is there a good reason to change the roles of x and y from the definition, where $H^s_{y\leftarrow x}$ was used?

39_8 "... dense in Σ for hyperbolic systems." - it is hard to understand what the author really means. It seems that Σ is a fixed mixing SFT from the beginning of section 3.4.

39_2 Is $\mathrm{per}(p)$ the minimal period of p?

40^4 Figure 10 appears in the thesis 30 pages later.

45^3 In assumptions of Theorem 3.4.18 we have $\alpha(\mathcal{A}) < \infty$. Then the conclusion of this theorem says that $\alpha(\mathcal{A})$ exists. What does it mean?

45_7 What is "Hausdorff dimension level set"?

46^2 What is "mixing subshift of finite type on compact metric space"?

46_{13} If it is easy to see, why not too include the calculation?

46_6 Are all the assumptions of Theorem 3.4.18 satisfied?

46_6 Can you be more specific how (and which version) of variational principle is applied here?

47_3 It is not the most optimal place to introduce notation, like interior or closure.

47_2 Is the statement that $\mathring{L}$ is convex a part of statement of Lemma 3.5.4?

48^1 In the statement of Lemma 3.5.4 we use L (defined on p.2) while Theorem 3.5.1 deals with $\overrightarrow{L}$. These sets are defined from different points of view. It should be clarified that they are the same here (and later).

48^5 I am not convinced by the proof of Theorem 3.5.5. First of all, we need to know that there is $h_{top}(E(\alpha)) > 0$ for some α to cover constant function case. Then we should use argument that minimum at 0 cannot be attained rather than using monotonicity argument.

49^7 "We start with..." - this fact was stated several pages back. Do we want to recall/emphasize it here?

49^{10} "Let (X, T) be..." - is there any good reason to not state these crucial assumptions in the statement of theorem? The setting changes from section to section, so it is easy to overlook it.

49_{10} Not sure if it is obvious. But follows by compactness.

49_9 "we have $\partial P(t) = ...$" - what happened with symbol Eq from Proposition 3.3.7?

49_4 "... are upper semi-continuous..." - can you provide a reference to theorem stating that. It is not directly assumed.

49_1 Remark ^{2} seems simple, can you explain it in detail?

50^8 Formally ∂^e can be a set. What is $\partial^e P(t_-)$? It is inconsistent with definitions on p.14.

50^{18} Do not see the definition. And depending on the way it is done, the inequality can follow from it directly or needs an equivalent description first.

51_1 Symbol (x) is missing, i.e. it seems a function $\phi_n(x)$ is defined here.

52^1 There is no "Anosov" in the definition of closing property.

52^4 How do we know that "ergodic maximizing measure" exists? Provide a reference.

52^5 Since x is fixed, $\mu_{n,x}$ is uniquely defined, not "exists" by a proper choice. Convergence to μ is by the choice of x as well. Some comment would be appreciated here.

52^{10} "Let $p \in X$ be a periodic..." - first of all, it should be p_n. Second, it does not exist for every n. There is an important role of δ in the definition of closing property (see p.27).

52^{13} Lemma 3.5.15 is a general fact, which should be proved before 3.5.14. Its statement is very vague. Should be more precise, with some extra constants. The proof is also very loosely written. Definitely all the constants δ, C, ε should be applied with exact calculations instead of formulations like "very close".

52_8 "By the Anosov closing property..." - was there any reason to spend time on proving Lem 3.5.15? It seems, it was never used in the proof.

52_7 How assumptions of Thm. 3.4.12 and and (3.2.1) are satisfied? What cocycles $\mathcal{A}_n$ are used? The same comment applies to application of Thm. 3.4.11. There is no Lemma 4.3.4! And we are in section 3.

52_5 Isn't S_n missing in (3.5.4.1)? Note that orbit of p is close to x only during first n iterations. The same problem appears in 52^8. It seems integral is replaced by a sum.

52_1 What is Theorem [BBB]? Is it Theorem 3.4.12 used already in the proof of Thm. 3.5.14?

63^7 It suggest changing $||v|$ to : $|v|$

63_8 Why 2π is there? Is $\eta : M \to \mathbb{R}$? How are functions u, v defined?

63_7 What does $\times$ mean? Is it just a multiplication $d \cdot x$? What does it mean 2π-periodic with respect to x? We have $x \in S^1$ not $x \in \mathbb{R}$.

63_5 What degree do we mean here? M is 3-dim manifold. There is also a collision between d as degree and d as a metric at top of the page.

63_3 There should be $\phi = \eta$ not $\phi = \eta'$. Symbol $'$ should appear in the definition of ϕ_n. Then it is consistent with later formula on λ_n.

63_{10} Why "1-" not "1."?

64^{8} Is "$f|_\Lambda$ transitive" an assumption? A consequence of some theorem? Is dominated splitting assumed to exists? Or it exists as a consequence of some properties?

64^{12} Saying "ε small enough" in a definition is not very precise.

64_9 Is $\pi\colon (x,y,z) \mapsto x$? What is π_D? Sentence "For any set... let $p \in D_x$" is not clear. What is p? What is $W^s_{D_x}(p)$? I do not see its definition in section 2.8. Are p and x related somehow?

64_6 Is there any common convention of symbols? Why $\pi_{(x,y)}$ but π_D? Why not use $\pi_{(x)}$?

64_3 Was transversal ever defined?

64^6 Is unstable lamination somehow related to unstable foliation?

64_5 How to read $|x(p)-x(q)| < 2\pi$ in case of S^1? Is not diam $S^1 = \pi$ by the definition (we use arc-lenght metric in S^1, aren't we)?

65_2 Do we assume "local product structure"?

66_{11} "Due to integrability..." - what does it exactly mean?

66_{10} Application of Thm. 4.1.3 is not clear. Sentence "That means $p = (x,y,z)$" is without sense. Is $W^s(p) \cap S^1$ a singleton? Why is L invertible? By the way, symbol L is reserved for Birkhoff spectrum.

66_5 Symbol f' is one of the worst choices. Why not $\hat{f}$? Same comment about h'_i. How do we get this decomposition of f'?

67^7 Are a, b_1, c_1 etc. functions? Constants? How exactly we get 0's in $D_e f$, in particular in the last column?

67^9 The paragraph "The leaves..." is completely unclear. How are these 1-1 projections to S^1 obtained? What does "angle" mean?

67^{14} The paragraph "Under transversality..." is unclear as well. What is μ_t and more importantly, what is $\chi(\mu_t, \eta')$? The map η' cannot be iterated.

67_{12} The set $\eta^{-1}(0)$ is a subset of M. What does it mean $I_i = [a_i, a_{i+1})$ and how can it be included in S^1?

86 Bibliography style needs some uniform structure. There are positions like [Bow], [B], [BP07], [Feng1], [1],... There are several standard ways of enumeration that Latex/Bibtex provides. The most appropriate style from the above seems [BP07], which should appear in other positions as e.g. [Ano67], [Moh20a], [Moh20b], etc. Also formating of bibliography entries should be improved. For example in [B] we have special underline for repeated author references, which is not used in other similar cases, e.g. in [Boc1]. What is 350 in [CQ]? In [FFW] or [MMR] bold font is missing, book references [F7] and [F8] have different style. In [Ka] should be (2) not (2), [GR] is missing ":", some journals are in abbreviated form, some other have full title, etc.