INSTITUTE OF MATHEMATICS
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# THE COMBINATORIAL STRUCTURE OF CUMULANTS OF SYMMETRIC FUNCTIONS

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# Declaration of authorship

I, Maciej Kowalski, declare that this thesis titled “The combinatorial structure of cumulants of symmetric functions” and the work presented in it are my own.

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I, Maciej Dołęga, declare that this thesis is ready for evaluation by reviewers.

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# Abstract

The thesis is divided into three parts. We begin by introducing the reader to the topic of symmetric functions, and later move on to study the main object of focus in the thesis: *cumulants*. We conclude by proving a special case of an open conjecture concerning LLT cumulants.

In the first part, which consists of two chapters, we define the space of symmetric functions, as well as introduce classical notions and results in the theory. We highlight the connections to other branches of mathematics, such as representation theory or algebraic geometry to give motivation to studying symmetric functions. Most importantly, we mention the conjecture of Dołęga on Schur positivity of Macdonald cumulants, which was the starting point behind the results presented in the thesis.

The second part introduces a new class of symmetric functions: *LLT cumulants*, which aim to serve as a means to prove the conjecture of Dołęga. We define different normalizations of the cumulants and prove that Macdonald cumulants have a positive expansion in terms of LLT cumulants of ribbon shapes. Also, we interpret LLT cumulants as a weighted generating function of certain graph colorings, which proves useful in the study of the cumulants and allows for proving a number of positivity results, generalizing several recent advances in the field.

The last part gives a proof of a combinatorial formula for a positive LLT expansion of the LLT cumulant in the case when the sequence of shapes corresponds to a class of graphs called melting lollipops. In order to achieve that, we define Schröder paths and Schröder path relations which help understand the structure of unicellular LLT cumulants and, as a consequence, give an algorithmic way of decomposing LLT cumulants of melting lollipops.

# Streszczenie

Praca podzielona jest na trzy części. Rozpoczynamy od wprowadzenia czytelnika w temat funkcji symetrycznych, po czym przechodzimy do badania obiektów, na których skupiamy się w pracy: *kumulant*. Na koniec dowodzimy szczególny przypadek otwartej hipotezy dotyczącej kumulant LLT.

W pierwszej części, która składa się z dwóch rozdziałów, definiujemy przestrzeń funkcji symetrycznych, a także wprowadzamy klasyczne pojęcia i wyniki tejże teorii. Opisujemy związki z innymi działami matematyki, takimi jak teoria reprezentacji czy geometria algebraiczna, co daje dodatkową motywację do badań funkcji symetrycznych. W szczególności przywołujemy hipotezę Dołęgi dotyczącą Schur dodatniości kumulant Macdonalda, która była punktem startowym dla wyników prezentowanych w tejże pracy.

Druga część wprowadza nową klasę funkcji symetrycznych: *kumulanty LLT*, które mogą być narzędziem do dowodu hipotezy Dołęgi. Definiujemy różne normalizacje tychże kumulant i dowodzimy, że kumulanty Macdonalda wyrażają się jako dodatnia suma kumulant LLT kształtów taśmowych. Ponadto interpretujemy kumulanty LLT jako ważoną funkcję tworzącą pewnych pokolorowań grafów, co okazuje się pomocne w badaniu kumulant i pozwala dowieść kilku wyników o dodatniości, rozszerzając między innymi niedawne wyniki w tej dziedzinie.

Ostatnia część prezentuje dowód kombinatorycznej formuły na dodatnie rozwinięcie kumulant LLT w przypadku, gdy ciąg kształtów należy do klasy grafów zwanych topniejącymi lizakami. W tym celu badamy ścieżki Schrödera i relacje na tychże ścieżkach, które pomagają zrozumieć strukturę jednokomórkowych kumulant LLT i w konsekwencji dają algorytmiczny sposób na rozkład kumulant LLT topniejących lizaków.

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# Chapter 1

# Introduction

The first results dedicated to symmetric functions date back as far as the 17th century. While not yet a field of their own at the time, they soon began appearing in different, seemingly unrelated contexts, which later allowed for connecting distinct branches of mathematics (see Section 1.1) into the vast theory that we have today. As such, a problem stated in the language of symmetric functions can often be translated into, e.g., representation theory or algebraic geometry.

The results presented in this thesis are motivated by one such problem: to understand the combinatorial structure of Macdonald cumulants and to extend the celebrated theorem of Haglund, Haiman, and Loehr [27] on the combinatorial formula for Macdonald polynomials to cumulants (see Theorem 2.2 for the precise formulation). They come from two papers: [12] and [42] (the first joint with the PhD supervisor, Maciej Dołęga) and are given in Chapter 3 and Chapter 4, respectively.

In this chapter, we define the expressions we focus on in the rest of the thesis, as well as give motivation for studying them and the problem at hand. Lastly, we describe the obtained results.

## 1.1 Symmetric functions and Schur functions

We will now formally introduce the objects of focus in the thesis: symmetric functions. We begin with the classical and basic definitions and move on to the generalizations in consecutive sections. Also, we highlight the connections to other branches of mathematics to display the motivations to study both symmetric functions as a whole, and the problems stated in the thesis in particular.

We say that a polynomial $f$ in variables $x_1, \ldots, x_n$ is *symmetric* if it is invariant under the permutation of variables, i.e., if $f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for all $\sigma \in \mathfrak{S}_n$. We can think of *symmetric functions* as the polynomials' equivalent in infinitely many variables (for a precise definition, see Chapter 2).

It turns out that the space $\Lambda$ of symmetric functions has a "nice" algebraic structure (see Chapter 2 for the detailed definition). Obviously, it is a linear space over any given field $K$ (in our case, $K = \mathbb{Q}$) but additionally, the natural polynomial multiplication in $\Lambda$ defines the structure of a ring. One algebraic question concerning $\Lambda$ which will be particularly important for us is that of finding a basis of the space. One can easily verify that as a linear space, it is spanned by polynomials of the form

$$m_\lambda \coloneqq \sum_\alpha x^\alpha,$$

where $\lambda = (\lambda_1, \lambda_2, \dots)$ with $|\lambda| \coloneqq \sum_i \lambda_i$ finite is a partition, i.e., a non-increasing sequence of non-negative integers, and we sum over all distinct permutations $\alpha$ of $\lambda$. We call $(m_\lambda)_\lambda$ the *monomial basis* of $\Lambda$.

**Example.** *We have*

$$m_{(1)} = \sum_{i \in \mathbb{Z}_{>0}} x_i, \qquad m_{(2)} = \sum_{i \in \mathbb{Z}_{>0}} x_i^2, \qquad m_{(1,1)} = \sum_{i<j} x_i x_j.$$

The monomials are clearly not the only basis of $\Lambda$ and it turns out that different bases have their own motivation to study them. Let us now introduce a family of functions that will be of particular interest to us in the remainder of the thesis due to its close connections with other areas of mathematics.

**Definition 1.1.** For a partition $\lambda = (\lambda_1, \dots, \lambda_n, 0, \dots)$, we define the *Schur function* $s_\lambda$ to be the expression

$$s_\lambda(x_1, \dots, x_n) = \frac{a_{\lambda+\delta_n}}{a_{\delta_n}},$$

where $a_\alpha = \det(x_j^{\alpha_i})$, and $\delta_n = (n-1, n-2, \dots, 1, 0)$. Additionally, for a partition $\mu$ with $\mu_i \leq \lambda_i$ for $i > 0$, we define the *skew Schur function* $s_{\lambda/\mu}$ to be the function satisfying $\langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\lambda, s_\mu s_\nu \rangle$, where $\langle \cdot, \cdot \rangle$ is the Hall inner product, for which $\langle s_\nu, s_\tau \rangle = \delta_{\nu,\tau}$.

Schur functions first appeared in 1815 in the works of Cauchy, although not under that name and under a different (though equivalent) definition. While the notion attracted attention from the purely algebraic point of view, the functions independently appeared in the field of algebraic geometry in the works of Schubert (see [39] for references) in, what is now called, *Schubert calculus*. In particular, Giambelli [23] showed the geometric equivalent of the Jacobi-Trudi identity [35]: a formula that relates different bases of $\Lambda$ with each other (see Theorem 1.3), hence introducing the equivalence between symmetric functions and algebraic geometry. What is more, symmetric functions are also present in representation theory of the symmetric group or the general linear group. There, Schur functions are the characters of finite-dimensional irreducible representations of $\mathrm{GL}_n$ [55] and thus, we obtain another link between different areas of mathematics.

In fact, Schur himself studied what we now call by his name in relation with representation theory, in particular in connection with the symmetric group [55]. There, he managed to use combinatorics to answer the natural question of expressing Schur functions in terms of the monomial basis, thanks to which we obtain a beautiful combinatorial interpretation of the coefficients in the expansion. However, before we present the formula (i.e., Theorem 1.1), let us introduce a few definitions.

The *Young diagram of shape* $\lambda$ is the arrangement of $|\lambda|$ $1\times 1$ boxes with $\lambda_1$ boxes in the bottom row, $\lambda_2$ in the second, etc., all aligned to the bottom-left (i.e., we use the French notation for the diagrams). A *Young tableau* $T$ of shape $\lambda$ is a filling of the boxes with positive integers, to which we sometimes refer to as a map $T : \lambda \to \mathbb{Z}_{>0}$. Additionally, if the entries are increasing upwards and non-decreasing rightwards, we call the tableau *semistandard* and denote the fact by $T \in \mathrm{SSYT}(\lambda)$. A semistandard Young tableau of size $n$ (i.e., with $|\lambda| = n$) with entries 1 through $n$ is called *standard* (denoted $T \in \mathrm{SYT}(\lambda)$).

**Theorem 1.1.** For a partition $\lambda$, we have

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T,$$

where $x^T := \prod_{i \in \mathbb{Z}_{>0}} x_i^{\alpha_i(T)}$ for $\alpha_i(T)$ the number of occurrences of $i$ in $T$.

**Example.** *There are four distinct minimal (i.e., with the entries minimal while preserving the relations between them) semistandard Young tableaux of shape* $(2,1)$*:*

$$\begin{array}{|c|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline \end{array} \qquad \begin{array}{|c|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline \end{array} \qquad \begin{array}{|c|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{3} \\ \hline \end{array} \qquad \begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline \end{array}$$

*Thus, following Theorem 1.1, we have*

$$s_{(2,1)} = \sum_{i<j} (x_i^2 x_j + x_i x_j^2) + 2 \sum_{i<j<k} x_i x_j x_k.$$

In general, the use of Young tableaux extends far beyond the study of symmetric functions and they will be essential in the chapters that follow. For that reason and because of our focus on combinatorics in this thesis, we will treat Theorem 1.1 as the definition of Schur polynomials.

Let us also introduce three other bases of $\Lambda$. For all $k \in \mathbb{Z}_{>0}$, we define

$$e_k := \sum_{i_1 < \cdots < i_k} x_{i_1} \ldots x_{i_k}, \qquad h_k := \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k}, \qquad p_k := \sum_{i \in \mathbb{Z}_{>0}} x_i^k$$

and

$$e_\lambda := e_{\lambda_1} \cdots e_{\lambda_k}, \qquad h_\lambda := h_{\lambda_1} \cdots h_{\lambda_k}, \qquad p_\lambda := p_{\lambda_1} \cdots p_{\lambda_k}$$

for $\lambda$ a partition with $l(\lambda) = k$ (i.e., for $\lambda$ with $k$ non-zero elements). We call $(e_\lambda)_\lambda$, $(h_\lambda)_\lambda$, and $(p_\lambda)_\lambda$ the *elementary*, *complete*, and *powersum basis* of $\Lambda$, respectively (although it is non-trivial to show that they indeed generate $\Lambda$ as a linear space).

It is immediate to ask a similar question as before: how do we expand one of the bases in terms of the others? In fact, such a question is a common problem when studying symmetric functions and it is the end goal of the conjectures and partial results presented in the thesis. For example, for the basic functions defined above, we have

**Proposition 1.2.** For a partition $\lambda \vdash n$ (i.e., when $|\lambda| = n$), we have

1. $e_\lambda = \sum_{\mu \vdash n} M_{\lambda,\mu} m_\mu$, where $M_{\lambda,\mu}$ is the number of matrices with entries 0 or 1 whose $i$-th row and $j$-th column sum up to $\lambda_i$ and $\mu_j$, respectively;

2. $h_\lambda = \sum_{\mu \vdash n} N_{\lambda,\mu} m_\mu$, where $N_{\lambda,\mu}$ is the number of matrices with entries from $\mathbb{Z}_{\geq 0}$ whose $i$-th row and $j$-th column sum up to $\lambda_i$ and $\mu_j$, respectively;

3. $p_\lambda = \sum_{\mu \vdash n} R_{\lambda,\mu} m_\mu$, where $R_{\lambda,\mu}$ is the number of ordered set partitions $(B_1, \dots, B_{l(\mu)})$ of $[l(\lambda)]$, i.e., ordered tuples of sets with $\bigcup_i B_i = [l(\lambda)]$ (here and throughout, $[n] := \{1, 2, \dots, n\}$), for which $\sum_{b \in B_i} \lambda_b = \mu_i$ for $i = 1, \dots, l(\mu)$.

From Proposition 1.2, we immediately see that any function that has a positive expansion (i.e., an expansion with coefficients in $\mathbb{Z}_{>0}$) in the elementary, complete, or powersum basis will have a positive expansion in monomials. As such, we conclude that $e$-, $h$-, and $p$-positivity are stronger properties than $m$-positivity.

Similarly, one may wonder how we can express Schur polynomials in terms of the other bases to establish a hierarchy between different positivity properties. On the one hand, going from elementary or complete basis to Schur polynomials comes from a simple observation that $e_k = s_{(1^k)}$ and $h_k = s_{(k)}$ and the Pieri rule: a formula describing the product of $s_\lambda$ with $e_k$ or $h_k$. On the other hand, the converse relation is a non-trivial formula due to Jacobi and Trudi [35].

**Theorem 1.3** (Jacobi-Trudi)**.** [35] For a partition $\lambda$ with $k$ non-zero elements,

$$s_\lambda = \det(h_{\lambda_i - i + j}) = \det(e_{\lambda^t_j + i - j}),$$

where $\lambda^t$ denotes the *transpose* of $\lambda$ (see Chapter 2) and the matrices are of size $k \times k$ (we follow the convention that negative index corresponds to 0).

In fact, Schur positivity is of particular interest in the general study of symmetric functions and the same applies in our particular case, i.e., in the study of symmetric function cumulants (see Section 1.3). It is a consequence of the close connections between different areas of mathematics that we have presented above and the multiple appearances of Schur functions in the formulas. In general, the language of symmetric functions proves universal in problems stated in multiple contexts and allows the use of combinatorics or general algebra to obtain results in representation theory or algebraic geometry. What is more, the converse is also true: we may apply the language of representation theory or algebraic geometry to prove interesting combinatorial formulas.

In particular, there is a straightforward connection between representation theory and symmetric functions. To be precise, we know that every $S_n$-module $V$ can be decomposed into its irreducible submodules $V = \bigoplus_\lambda V_\lambda^{c_\lambda}$ and we can define the *Frobenius characteristic* as the map $\mathrm{Fr} : V_\lambda \longmapsto s_\lambda$ that assigns a Schur function to every irreducible submodule. This map extends to direct sums so in particular, if one can show that a given function is a Frobenius characteristic of some $S_n$-module, then it must be Schur positive.

For example, one can show that the regular representations correspond via the above map to the product $(m_{(1)})^n$. What is more, it turns out that we have a beautiful combinatorial interpretation of its Schur expansion:

$$m_{(1)}^n = \sum_{\lambda \vdash n} |\{T \mid T \in \mathrm{SYT}(\lambda)\}| s_\lambda.$$

We want to stress that the combinatorics is precisely what we focus on in the thesis and that we did not attempt to translate our results to other fields: a task that may be challenging in itself but certainly worthwile.

## 1.2 Macdonald polynomials and LLT polynomials

Across the years, mathematicians have studied several generalizations of Schur functions, often by introducing a parameter that tracks a given property or statistic on the tableaux. In 1961, in the context of Hall algebras, a new family of symmetric functions were introduced: $P_\lambda(t)$ dependent on a partition $\lambda$ and a parameter $t$, which were later directly defined by Littlewood and are now known under the name *Hall-Littlewood polynomials* [49]. In particular, we have $P_\lambda(1) = m_\lambda$ and $P_\lambda(0) = s_\lambda$. In 1970, Jack [34] introduced symmetric functions $J_\lambda^{(\alpha)}$ indexed by a partition $\lambda$ and a parameter $\alpha$, which were later called *Jack polynomials*. In that case, we have $J_\lambda^{(1)} = B_\lambda s_\lambda$ for some scalar $B_\lambda$. Lastly, in 1997, while studying quantum groups, Lascoux, Leclerc, and Thibon [45] defined what we now call *LLT polynomials* (see Definition 2.1 for the formal definition) which we denote here by $\mathrm{LLT}(q)$: a $q$-deformation of a product of Schur functions.

With the above in mind, Macdonald [50] aimed to introduce polynomials that would generalize all the newly appearing symmetric functions[1]. He introduced a family of two-parameter symmetric functions which are now known as *Macdonald polynomials* (see Theorem 2.2 for the definition) and conjectured that when expanded in the basis of Schur symmetric functions, the coefficients are polynomials in two deformation parameters $q, t$ with nonnegative integer coefficients.

**Example.** *We have the following Schur expansions of Macdonald polynomials corresponding to all three shapes $\lambda$ with $|\lambda| = 3$:*

$$\begin{aligned}
\tilde{H}^{(q,t)}_{(1,1,1)} &= t^3 s_{(1,1,1)} + (t^2+t) s_{(2,1)} + s_{(3)},\\
\tilde{H}^{(q,t)}_{(2,1)} &= qt s_{(1,1,1)} + (q+t) s_{(2,1)} + s_{(3)},\\
\tilde{H}^{(q,t)}_{(3)} &= q^3 s_{(1,1,1)} + (q^2+q) s_{(2,1)} + s_{(3)}.
\end{aligned}$$

**Conjecture 1** (Macdonald [50])**.** *For any partions $\lambda$ and $\mu$, we have*

$$[s_\mu]\tilde{H}^{(q,t)}_\lambda \in \mathbb{Z}_{\geq 0}[q,t].$$

[1]Note that LLT polynomials were, in fact, introduced later than Macdonald polynomials and are defined more generally, i.e., for sequences of shapes. Nevertheless, their connection to Macdonald polynomials quickly became apparent and is extremely important in the context of this thesis (see Section 2.2.3).

Conjecture 1 was indeed proved by Haiman [29] using the Frobenius characteristic argument that we have presented in Section 1.1: Haiman showed that the modified Macdonald polynomials are the Frobenius characteristic of the linear spans of certain determinants in variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ that naturally carry the action of the symmetric group.

Nevertheless, given Haiman's result, it would be most desirable to find a combinatorial interpretation of the Schur expansion, i.e., to define characteristics on the space of Young tableaux that would describe the exponents of $q$ and $t$ in the polynomial coefficients of $s_\lambda$ in the expansion. It turned out that the attempts to find it resulted in a huge development of the field.

In 1995 Lapointe and Vinet [43] proved that the coefficients of Jack symmetric functions expanded in the monomial basis are polynomials in the deformation parameter $\alpha$ with integer coefficient. Two years later, Knop and Sahi found an explicit positive formula for this expansion [41]. Since Jack symmetric functions are a limit case of Macdonald symmetric functions, these results inspired further research and shortly afterwards, the polynomiality of the coefficients of Macdonald polynomials was proved independently and almost simultaneously (using different approaches) in five different papers [53, 20, 44, 40, 38]. An affirmative answer to Macdonald's original conjecture was finally released in a difficult paper of Haiman [29], who was able to relate Macdonald's question with a question about the geometry of Hilbert schemes of points in the complex plane, to which he gave an affirmative answer. Even though this result built new bridges between various fields of mathematics, it did not provide an explicit combinatorial formula explaining Schur-positivity and the conjecture remains open.

In 2005, Haglund, Haiman and Loehr found an explicit combinatorial formula for Macdonald polynomials [27], lifting Knop and Sahi's formula to the two-parameter world of Macdonald, and relating Macdonald polynomials with LLT polynomials. Haglund, Haiman and Loehr noticed in [27] that Macdonald polynomials can be naturally decomposed as a positive combination of their LLT counterparts, so that proving Schur-positivity for LLT polynomials would give yet another proof of the famous conjecture of Macdonald. This was done by Grojnowski and Haiman [25], who related LLT polynomials with the Kazhdan-Lusztig theory in a much more general setting than it was done before [46], and therefore proved the Schur-positivity of LLT polynomials indexed by arbitrary skew-shapes (see Section 2.2 for the necessary definitions).

## 1.3 Symmetric function cumulants

The notion of cumulants was originally studied by Leonov and Shiryaev [48] in the context of probability theory. Cumulants appear now in a wide variety of contexts, see [36, Chapter 6] for their role in studying random graphs and [51] for a concise introduction to noncommutative probability theory and various types of cumulants.

In 2017, Dołęga and Féray [15] introduced *Jack cumulants* as a tool to approach a famous open problem in the theory of symmetric functions known as *the b-conjecture* (and posed by Goulden and

Jackson [24]), which relates Jack symmetric functions with a weighted generting function of graphs embedded into surfaces (and which, despite some recent progress [9], is still open). The notion of Jack cumulants naturally extends to *Macdonald cumulants* the same way as Jack polynomials can be seen as the limit case of Macdonald polynomials (in the thesis, we do not consider Jack polynomials or Jack cumulants so we refer the interested reader to [34] for more information on the subject).

Dołęga and Féray observed conjecturally that the coefficients of Macdonald cumulants seem to be polynomials, which was later proved in [13] and further improved in [14], where an explicit positive combinatorial formula for the Macdonald cumulants was proved. This rich combinatorial structure of Macdonald cumulants naturally calls for investigating the expansion in the Schur basis: extensive computer simulations performed by Dołęga [13] have led him to believe that a more general version of the original question of Macdonald is true: the coefficients of the Schur expansion of Macdonald cumulants are polynomials in $q, t$ with nonnegative integer coefficients (see Conjecture 2).

On the one hand, Hausel, Letellier and Rodriguez Villegas [30] considered the logarithm of a partition function for Macdonald polynomials and conjectured its monomial positivity interpreted as the mixed Hodge polynomials of character varieties. On the other hand, the notion of Macdonald cumulants appears naturally in the decomposition of the logarithm of the partition function, and the recent work [1] (following [31, 10]) exhibits that the Poincaré polynomial of Nakajima quiver variaties (which can be seen as a special case of the aforementioned conjecture) is given by the specialization of the Tutte polynomial $\mathrm{Tutte}_G(1, q)$, which is the same phenomenon as in Dołęga's combinatorial interpretation of Macdonald cumulants [14]. All this gives yet additional motivation for studying the combinatorial structure of Macdonald cumulants.

In what follows, our focus lies on the $q$-deformation of partial cumulants that appeared in [13] and was inspired by the classical definition of conditional cumulants (see Section 3.1).

**Definition 1.2.** Suppose that $\mathcal{A}$ is an algebra over the fraction field $\mathbb{Q}(q)$. Let $\boldsymbol{u} := (u_I)_{I \subseteq V}$ be a family of elements in $\mathcal{A}$, indexed by subsets of a finite set $V$. Then for any non-empty subset $I$ of $V$, the *$q$-partial cumulant* of $\boldsymbol{u}$ corresponding to $I$ is the expression

$$\kappa_I^{(q)}(\boldsymbol{u}) := (q-1)^{1-|I|} \sum_{\pi \in \mathcal{P}(I)} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{B \in \pi} u_B. \tag{1.1}$$

The sum runs over all elements of the family $\mathcal{P}(I)$ of *set partitions* of $I$: its elements are sets of disjoint subsets of $I$ whose union is equal to $I$ (so one can think that an element $\pi \in \mathcal{P}(I)$ is grouping elements of $I$ into disjoint subsets).

**Example.** *For* $V = \{1, 2, 3\}$*, we have:*

$$\kappa_{\{1\}}^{(q)}(\boldsymbol{u}) = u_{\{1\}},$$
$$\kappa_{\{1,2\}}^{(q)}(\boldsymbol{u}) = (q-1)^{-1}(u_{\{1,2\}} - u_{\{1\}}u_{\{2\}}),$$
$$\kappa_{\{1,2,3\}}^{(q)}(\boldsymbol{u}) = (q-1)^{-2}(u_{\{1,2,3\}} - u_{\{1\}}u_{\{2,3\}} - u_{\{2\}}u_{\{1,3\}} - u_{\{3\}}u_{\{1,2\}} + 2u_{\{1\}}u_{\{2\}}u_{\{3\}}).$$

In the thesis, we are mostly interested in the case when $\boldsymbol{u}$ is a family of symmetric functions and thus, when $\kappa_I^{(q)}(\boldsymbol{u})$ are symmetric functions as well. We will expand on this idea in the chapters that follow, especially in Chapter 3.

## 1.4 Structure of the thesis

In Chapter 2, we formally introduce the basic notions that we use throughout the thesis. We begin with the definition of the space of symmetric functions, followed by Schur functions, and concluded by LLT and Macdonald polynomials.

Chapter 3 focuses on the study of LLT cumulants and its connections to the conjecture on Schur positivity of Macdonald cumulants. In particular, we define the cumulants in a more general setting, as well as the polynomial cumulants themselves and we prove that Macdonald cumulants have an explicit positive expansion in terms of LLT cumulants of ribbon shapes. Also, we introduce the notion of an LLT graph and express LLT cumulants as a weighted generating function of graph colorings. We conclude by applying the graph theoretical framework to prove various positivity results.

In Chapter 4, we prove a combinatorial formula for a positive LLT expansion of the LLT cumulant in a special case: when the LLT graph is a melting lollipop. In particular, this proves that those cumulants are Schur positive. For the proof, we use the notion of Schröder paths and Schröder path relations, which we introduce in one of the sections. We finish the chapter by applying the result to the complete graph case.

# Chapter 2

# Preliminaries

In this chapter, we introduce the formal definition of the space of symmetric functions as well as introduce some basic notions connected with it. Also, we define LLT and Macdonald polynomials and describe the relation between them.

## 2.1 The space of symmetric functions

Let $\mathbb{Q}[x_1, \ldots, x_n]$ denote the space of polynomials in $n$ variables over $\mathbb{Q}$ (in general, we can take any field $K$, but in the thesis, $\mathbb{Q}$ suffices). Define the action of the symmetric group $\mathfrak{S}_n$ on $\mathbb{Q}[x_1, \ldots, x_n]$ by

$$\sigma(f(x_1, \ldots, x_n)) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

for $\sigma \in \mathfrak{S}_n$ and $f \in \mathbb{Q}[x_1, \ldots, x_n]$. The elements of the subring $\Lambda_n := \mathbb{Q}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$ of polynomials invariant under the above action (i.e., invariant for every permutation $\sigma \in \mathfrak{S}_n$) are called *symmetric polynomials*.

Clearly, for $m \geq n$ we have a homomorphism $p : \Lambda_m \to \Lambda_n$ defined by

$$p(f(x_1, \ldots, x_m)) = f(x_1, \ldots, x_n, 0, \ldots, 0).$$

Additionally, for every $k$ and $n$, we can define by $\Lambda_n^k$ the subspace of polynomials in $\Lambda_n$ that are homogeneous of degree $k$. To be precise, $\Lambda_n^k$ consists of expressions in which monomials take the form $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha_1 + \cdots + \alpha_n = k$. Then, we can construct the space of *homogeneous symmetric functions* of degree $k$ by

$$\Lambda^k := \varprojlim_n \Lambda_n^k.$$

Finally, we define the *space of symmetric functions* to be

$$\Lambda := \bigoplus_{k \geq 0} \Lambda^k.$$

We note that the space of all symmetric functions is an algebra over $\mathbb{Q}$ with the standard operations of addition and multiplication.

### 2.1.1 Schur functions

In Chapter 1, we mentioned that Schur functions proved to be of great importance in multiple areas of mathematics. Most importantly, in representation theory, they describe the characters of finite-dimensional irreducible representations of the general linear group. Together with the fact that they form a basis of $\Lambda$, this leads us to a standard problem in the theory of symmetric functions: to find an expansion of a given family of symmetric functions in terms of Schur polynomials.

Oftentimes, the approach to such a question is to prove that the coefficients of such an expansion count some classical combinatorial objects. For instance, we have the following

**Theorem 2.1** (Littlewood-Richardson rule)**.** Let $\lambda$ and $\mu$ be partitions. Then

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu,$$

where $c^\nu_{\lambda,\mu}$ is the number of Littlewood-Richardson tableaux of shape $\nu/\lambda$ and weight $\mu$ (we refrain from defining the said objects since they are not used in the remainder of the thesis).

Note that the special case of the Littlewood-Richardson rule when $\mu = (1)^k$ or $\mu = (r)$, i.e., when $s_\mu = e_k$ or $s_\mu = h_k$, is known as the Pieri's formula or its dual version, respectively.

## 2.2 LLT polynomials and Macdonald polynomials

In our case, we will be aiming to find the Schur expansion of functions whose coefficients are polynomials in $q$: a parameter whose exponent will be described by a statistic on the combinatorial objects we study. In this context, Schur-positivity will mean that the coefficients of the Schur expansion belong to $\mathbb{Z}_{>0}[q]$.

### 2.2.1 LLT polynomials

For partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ with $\mu_i \leq \lambda_i$ for all $i > 0$, we define the *skew diagram of shape* $\lambda/\mu$ to be the set difference between the boxes of $\lambda$ and $\mu$. Let $\boldsymbol{\lambda}/\boldsymbol{\mu} = (\lambda^1/\mu^1, \dots, \lambda^m/\mu^m)$ be a sequence of skew partitions[1]. We say that a cell $\square = (x, y)$ (i.e., the box in row $y$ and column $x$) of $\lambda^i/\mu^i$ has *content* $c(\square) := x - y$ and *shifted content* $\tilde{c}(\square) := mc(\square) + i$. Lastly, for a box $\square \in \boldsymbol{\lambda}/\boldsymbol{\mu}$, we denote by $\square_\leftarrow$, $\square_\rightarrow$, $\square_\uparrow$, and $\square_\downarrow$ the boxes which are lying directly to the left, right, up, and down of the box $\square$, respectively, if such exist.

Let $\boldsymbol{T} \in \mathrm{SSYT}(\boldsymbol{\lambda}/\boldsymbol{\mu})$, where $\mathrm{SSYT}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ denotes the set of semistandard Young tableaux on $\boldsymbol{\lambda}/\boldsymbol{\mu} = (\lambda^1/\mu^1, \dots, \lambda^m/\mu^m)$. The set of *inversions* in $\boldsymbol{T}$ is defined to be

$$\mathrm{Inv}(\boldsymbol{T}) := \{(\square, \square') \in \boldsymbol{\lambda}/\boldsymbol{\mu} \mid 0 < \tilde{c}(\square') - \tilde{c}(\square) < m \text{ and } \boldsymbol{T}(\square) > \boldsymbol{T}(\square')\}.$$

We denote $\mathrm{inv}(\boldsymbol{T}) := |\,\mathrm{Inv}(\boldsymbol{T})|$.

[1]Note how from now on, we use bolded notation for sequences of shapes and regular font for single shapes.

**Definition 2.1.** The *LLT polynomial* of $\boldsymbol{\lambda}/\boldsymbol{\mu}$ is the generating function

$$\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu}) = \mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu})(q) := \sum_{\boldsymbol{T}\in\mathrm{SSYT}(\boldsymbol{\lambda}/\boldsymbol{\mu})} q^{\mathrm{inv}(\boldsymbol{T})} x^{\boldsymbol{T}}. \tag{2.1}$$

If $\boldsymbol{\lambda}/\boldsymbol{\mu}$ is a sequence of unicellular shapes, then $\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ is called a *unicelullar LLT polynomial.*

In particular, note that

$$\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu})(1) = \prod_{i=1}^{m} s_{\lambda^i/\mu^i}.$$

Definition 2.1 was introduced in [27] and it is related to the original definition of Lascoux, Leclerc and Thibon [45, Equation (26)] by:

$$\mathrm{LLT}^{\mathrm{cospin}}(\boldsymbol{\lambda}/\boldsymbol{\mu}) = q^{-\min_{\boldsymbol{T}\in\mathrm{SSYT}(\boldsymbol{\lambda}/\boldsymbol{\mu})}\mathrm{inv}(T)} \sum_{\boldsymbol{T}\in\mathrm{SSYT}(\boldsymbol{\lambda}/\boldsymbol{\mu})} q^{\mathrm{inv}(T)} x^{\boldsymbol{T}}. \tag{2.2}$$

**Remark 1.** *With the notation as above, we have* $\mathrm{LLT}^{\mathrm{cospin}}(\boldsymbol{\lambda}/\boldsymbol{\mu}) = \tilde{G}^{(r)}_\nu(X;q)$ *using the language from [45, Equation (26)]. We will now briefly describe the original definition of* $\tilde{G}^{(r)}_\nu(X;q)$ *and relate the reader to [45] for details.*

*For a partition* $\nu$ *and an integer* $r \in \mathbb{Z}_{>0}$*, take a sequence of shapes* $\nu = \nu^{(0)} \supseteq \nu^{(1)} \supseteq \cdots \supseteq \nu^{(l)}$ *such that the following conditions are satisfied:*

1. *for each* $i$*, the shape* $\nu^{(i)}/\nu^{(i+1)}$ *can be divided into* $r$*-ribbons (i.e., connected skew shapes of size* $r$ *without a* $2\times 2$ *square) in such a way that for each such* $r$*-ribbon and its bottom right cell* $\square$*, we have* $\square_\downarrow \notin \nu^{(i)}$ *(in fact, one can show that such a division is unique),*

2. *the sequence is maximal, i.e., we cannot define a shape* $\nu^{(l+1)}$ *for which* $\nu^{(l)}/\nu^{(l+1)}$ *is an* $r$*-ribbon.*

*It can be shown that while the integer* $l$ *can differ between different sequences, the last shape* $\nu^{(l)}$ *does not depend on the sequence.*

*We say that a filling* $T : \nu \to \mathbb{Z}_{>0}$ *of* $\nu$ *is an* $r$-ribbon tableau *if* $T$ *is constant on* $\nu^{(i)}/\nu^{(i+1)}$ *for each* $i$ *and* $T(\square) < T(\square')$ *whenever* $\square \in \nu^{(i)}$ *and* $\square' \in \nu^{(j)}$ *with* $i<j$*. Next, for each* $r$*-ribbon* $R$*, we define its* spin *(denoted* $\mathrm{spin}(R)$*) to be its number of rows minus* $1$ *and define the spin of* $T$ *to be* $\mathrm{spin}(T) := \sum \mathrm{spin}(R)$*. Lastly, we define the* cospin *of* $T$ *to be*

$$\mathrm{cospin}(T) := \frac{1}{2}\left(\max_S \mathrm{spin}(S) - \mathrm{spin}(T)\right),$$

*where the maximum is taken over all* $r$*-ribbon tableaux of* $\nu$*.*

*The* LLT polynomial $\tilde{G}^{(r)}_\nu(X;q)$ *is the generating function given by the formula*

$$\tilde{G}^{(r)}_\nu(X;q) := \sum_T q^{\mathrm{cospin}(T)} x^T, \tag{2.3}$$

*where we sum over all* $r$*-ribbon tableaux of* $\nu$*.*

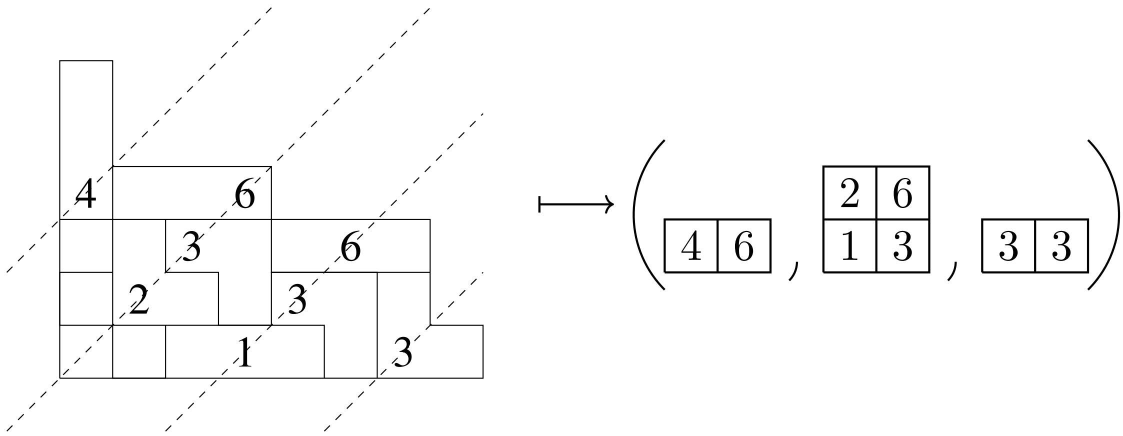


Figure 2.1: An example of the Stanton–White algorithm for a 3-ribbon tableaux of a diagram $(8,7,7,4,1,1)$.

*The relation between Definition 2.1 and Eq.* (2.3) *given in Eq.* (2.2) *is obtained via the Stanton–White algorithm [56], which we now briefly describe. To go from $\nu$ to $\boldsymbol{\lambda}/\boldsymbol{\mu}$, for each $r$-ribbon tableau $T$, we draw every $r$-th diagonal of $\nu$, i.e., a line going through cells at positions $(ir+j, j)$ for some $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_{>0}$. It is easy to see that each $r$-ribbon $R$ of $T$ intersects with exactly one of the diagonals (see Fig. 2.1).*

*Order the cells of an $r$-ribbon $R$ starting with the bottom right one and denote by $s(R)$ the index in this order of the cell intersected by one of the diagonals. We form $\boldsymbol{\lambda}/\boldsymbol{\mu}$ corresponding to $\nu$ by mapping $R$ of $T$ to a cell $\square \in \lambda^{(s(R))}/\mu^{(s(R))}$ in such a way that the position of the ribbons reflects the positions of the cells of $\boldsymbol{\lambda}/\boldsymbol{\mu}$. To be precise, a pair $(\square, \square_{\rightarrow})$ (respectively $(\square, \square_{\uparrow})$) of cells in $\boldsymbol{\lambda}/\boldsymbol{\mu}$ correspond to a pair of $r$-ribbons in $T$ with the first to the left (respectively, below) of the latter (see Fig. 2.1 for an example). It can be shown that the shape of $\boldsymbol{\lambda}/\boldsymbol{\mu}$ does not depend on the choice of $T$.*

### 2.2.2 The relation between different statistics on shape sequences

The statistic $\operatorname{inv}(T) - \min_{S\in \mathrm{SSYT}(\boldsymbol{\lambda}/\boldsymbol{\mu})} \operatorname{inv}(S)$ can be realized as the cardinality of a subset $\operatorname{Inv}_{\mathrm{cospin}}(T)$ of $\operatorname{Inv}(T)$ due to [54] (in particular, $\min_{T\in \mathrm{SSYT}(\boldsymbol{\lambda}/\boldsymbol{\mu})} \operatorname{inv}(T) = |\operatorname{Inv}(T) \setminus \operatorname{Inv}_{\mathrm{cospin}}(T)|$ for any $T \in \mathrm{SSYT}(\boldsymbol{\lambda}/\boldsymbol{\mu})$).

Define $\operatorname{Inv}_{\mathrm{cospin}}(T)$ as follows:

$$\begin{aligned}\operatorname{Inv}_{\mathrm{cospin}}(T) = \{(\square, \square') \in \operatorname{Inv}(T) : (\square'_{\uparrow}, \square) \in \operatorname{Inv}(T) \text{ and the row coordinate} \\ \text{of } \square \text{ is weakly smaller than the row coordinate of } \square'\}.\end{aligned}$$

Here, the convention is that for $\square'_{\uparrow} \notin \boldsymbol{\lambda}/\boldsymbol{\mu}$, the pair $(\square'_{\uparrow}, \square)$ is automatically an inversion. Then

$$\mathrm{LLT}^{\mathrm{cospin}}(\boldsymbol{\lambda}/\boldsymbol{\mu}) = \sum_{T\in \mathrm{SSYT}(\boldsymbol{\lambda}/\boldsymbol{\mu})} q^{|\operatorname{Inv}_{\mathrm{cospin}}(T)|} x^T. \tag{2.4}$$

In the special case when $\boldsymbol{\lambda}/\boldsymbol{\mu} = (\lambda^1/\mu^1, \ldots, \lambda^m/\mu^m)$ is a sequence of ribbon shapes, i.e., connected skew shapes which do not contain a shape of size $2 \times 2$, we define the normalization

$$\mathrm{LLT}^{\mathrm{Mac}}(\boldsymbol{\lambda}/\boldsymbol{\mu}) = q^{-a(\boldsymbol{\lambda}/\boldsymbol{\mu})} \sum_{T \in \mathrm{SSYT}(\boldsymbol{\lambda}/\boldsymbol{\mu})} q^{\mathrm{inv}(T)} x^T, \tag{2.5}$$

where

$$a(\boldsymbol{\lambda}/\boldsymbol{\mu}) = \sum_{\square \in \mathrm{Des}(\boldsymbol{\lambda}/\boldsymbol{\mu})} |\{\square' \mid c(\square') = c(\square), \tilde{c}(\square') > \tilde{c}(\square)\}|,$$

with $\mathrm{Des}(\boldsymbol{\lambda}/\boldsymbol{\mu}) = \{\square \in \boldsymbol{\lambda}/\boldsymbol{\mu} \mid \square_{\downarrow} \in \boldsymbol{\lambda}/\boldsymbol{\mu}\}$.

**Example.** *For $\boldsymbol{\lambda}/\boldsymbol{\mu} = ((1,1),(1),(1,1))$, we have*

$$\begin{aligned}
\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu}) &= q^2\big(q^4 s_{(1^5)} + (q^3+q^2)s_{(2,1^3)} + (q^2+q)s_{(2^2,1)} + qs_{(3,1^2)} + s_{(3,2)}\big),\\
\mathrm{LLT}^{\mathrm{Mac}}(\boldsymbol{\lambda}/\boldsymbol{\mu}) &= q\big(q^4 s_{(1^5)} + (q^3+q^2)s_{(2,1^3)} + (q^2+q)s_{(2^2,1)} + qs_{(3,1^2)} + s_{(3,2)}\big),\\
\mathrm{LLT}^{\mathrm{cospin}}(\boldsymbol{\lambda}/\boldsymbol{\mu}) &= q^4 s_{(1^5)} + (q^3+q^2)s_{(2,1^3)} + (q^2+q)s_{(2^2,1)} + qs_{(3,1^2)} + s_{(3,2)}.
\end{aligned}$$

### 2.2.3 Macdonald polynomials

The original definition due to Macdonald [50] is as follows. Define the scalar product $\langle \cdot, \cdot \rangle_{q,t}$ on the space of symmetric functions over $\mathbb{Q}(q,t)$ by

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} \prod_{i \geq 1} (m_i! i^{m_i}) \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}},$$

where $m_i = m_i(\lambda)$ denotes the number of occurrences of $i$ in $\lambda$.

**Definition 2.2.** We define the *Macdonald polynomials* to be the unique family of symmetric functions $(P_\lambda(q,t))_\lambda$ for which

1. $\langle P_\lambda, P_\mu \rangle_{q,t} = 0$ for $\lambda \neq \mu$ and

2. $P_\lambda(q,t) = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu$, where $u_{\lambda\mu}$ are coefficients in $\mathbb{Q}(q,t)$ and the sum follows the *dominance order* on partitions, i.e., $\mu < \lambda$ if and only if $\mu_1 + \cdots + \mu_k \leq \lambda_1 + \cdots + \lambda_k$ for all $k \geq 1$.

However, the above definition is troublesome in applications. Thus, in the thesis, we will be considering the *modified Macdonald polynomials* $\tilde{H}_\nu^{(q,t)}$. In particular, the choice of the last normalization in Section 2.2.2 is motivated by the combinatorial formula of Haglund, Haiman and Loehr, for Macdonald polynomials $\tilde{H}_\nu^{(q,t)}$. It expresses $\tilde{H}_\nu^{(q,t)}$ as a sum of LLT polynomials indexed by $\nu_1$-tuples of shapes (and for our purposes, we treat the formula as the definition of Macdonald polynomials):

**Theorem 2.2.** [27] For any partition $\nu$, the following expansion holds true

$$\tilde{H}_\nu^{(q,t)} = \sum_{\boldsymbol{\lambda}/\boldsymbol{\mu}} t^{\mathrm{maj}(\boldsymbol{\lambda}/\boldsymbol{\mu})} \, \mathrm{LLT}^{\mathrm{Mac}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}, \tag{2.6}$$

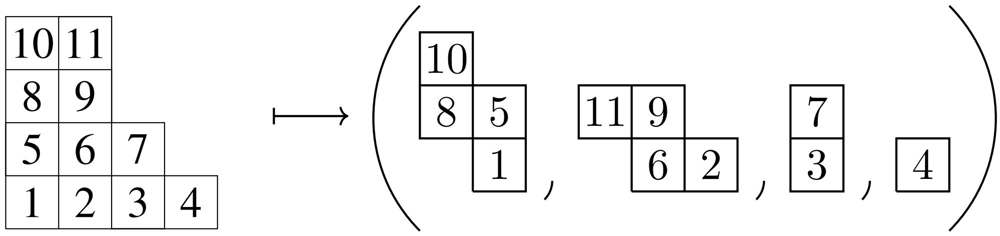


Figure 2.2: An example of a sequence of ribbons corresponding to the shape $(4,3,2,2)$ in the LLT expansion of the Macdonald polynomial according to Eq. (2.6) (the labels have been added solely to mark the cells corresponding to one another).

where we sum over all tuples of skew-partitions such that $\lambda^i/\mu^i$ is a ribbon of length $(\nu)^t_i$ whose bottom, far-right cell has content $0$.

The statistic $\mathrm{maj}$, which appears in (2.6), is defined as follows:

$$\mathrm{maj}(\boldsymbol{\lambda}/\boldsymbol{\mu}) := \sum_{i=1}^{\ell(\boldsymbol{\lambda}/\boldsymbol{\mu})} \mathrm{maj}(\lambda^i/\mu^i) = \sum_{i=1}^{\ell(\boldsymbol{\lambda}/\boldsymbol{\mu})} \sum_{\square \in \mathrm{Des}(\boldsymbol{\lambda}_i/\boldsymbol{\mu}_i)} |\{\square' \in \lambda^i/\mu^i \mid c(\square') < c(\square)| . \tag{2.7}$$

**Example.** *For $\nu = (4,3,2,2)$, an example summand in the LLT expansion given in Eq.* (2.6) *corresponds to the sequence $\boldsymbol{\lambda}/\boldsymbol{\mu} = ((3,3,2)/(2,1,1),(3,2)/(1),(3,3)/(2,2),(3)/(2))$ (see Fig. 2.2). For such a sequence, we have*

$$\mathrm{maj}(\boldsymbol{\lambda}/\boldsymbol{\mu}) = \sum_{i=1}^{l(\boldsymbol{\lambda}/\boldsymbol{\mu})} \mathrm{maj}(\lambda^i/\mu^i) = 2+1+0+0 = 3, \qquad a(\boldsymbol{\lambda}/\boldsymbol{\mu}) = 3.$$

# Chapter 3

# Cumulants and graph colorings

This chapter is a modified version of [12]: an article written by the author together with his PhD advisor, Maciej Dołęga.

**Abstract:** We introduce a new family of quasi-symmetric functions called LLT cumulants and discuss its properties. We define LLT cumulants using the algebraic framework for conditional cumulants and we prove that the Macdonald cumulant has an explicit positive expansion in terms of LLT cumulants of ribbon shapes, generalizing the classical decomposition of Macdonald polynomials. We also find a natural combinatorial interpretation of the LLT cumulant of a given directed graph as a weighted generating function of colorings of its subgraphs.

We use this graph theoretical framework to prove various positivity results. This includes monomial positivity, positivity in fundamental quasisymmetric functions and related positivity of the coefficients of Schur polynomials indexed by hook shapes. We also prove $e$-positivity for vertical-shape LLT cumulants, after the shift of variable $q \to q+1$, which refines a result of Alexandersson and Sulzgruber [5]. All these results give evidence towards Schur-positivity of LLT cumulants, which we conjecture here. We prove that this conjecture implies Schur-positivity of Macdonald cumulants, and we give more evidence by proving the conjecture for LLT cumulants of melting lollipops that refines a recent result of Huh, Nam and Yoo [32].

## 3.1 General cumulants

Even though we are most interested in cumulants in the context of symmetric functions, the notion is much more general. Therefore, we begin with the broad definitions which specialize to the cumulants mentioned in Section 1.3.

### 3.1.1 $q$-partial cumulants

The definition of partial cumulants is inspired by the classical definition of the *conditional cumulants*, which arose in the context of free probability theory (see, e.g., [48]). In what follows, we will be interested in the following $q$-deformation of partial cumulants.

**Definition 3.1.** Suppose that $\mathcal{A}$ is an algebra over the fraction field $\mathbb{Q}(q)$. Let $\boldsymbol{u} := (u_I)_{I \subseteq V}$ be a family of elements in $\mathcal{A}$, indexed by subsets of a finite set $V$. Then for any non-empty subset $I$ of $V$, the *$q$-partial cumulant* of $\boldsymbol{u}$ corresponding to $I$ is the expression

$$\kappa_I^{(q)}(\boldsymbol{u}) := (q-1)^{1-|I|} \sum_{\pi \in \mathcal{P}(I)} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{B \in \pi} u_B. \tag{3.1}$$

The sum runs over all elements of the family $\mathcal{P}(I)$ of *set partitions* of $I$: its elements are sets of disjoint subsets of $I$ whose union is equal to $I$ (so one can think that an element $\pi \in \mathcal{P}(I)$ is grouping elements of $I$ into disjoint subsets).

Indeed, in free probability theory, we have the following notion.

**Definition 3.2.** Let $\mathcal{A}$ be a vector space with two different commutative multiplicative structures $\cdot$ and $\oplus$, which define two (different) algebra structures on $\mathcal{A}$. For any $X_1, \dots, X_r \in \mathcal{A}$, we define the *conditional cumulant* $\kappa(X_1, \dots, X_r) \in \mathcal{A}$ as the coefficient of $t_1 \cdots t_r$ in the following formal power series in $t_1, \dots, t_r$:

$$\kappa(X_1, \dots, X_r) := [t_1 \cdots t_r] \log_{\cdot} \left(\exp_{\oplus}(t_1 X_1 + \cdots + t_r X_r)\right), \tag{3.2}$$

where $\log_{\cdot}$ and $\exp_{\oplus}$ are defined in the standard way with respect to multiplication given by $\cdot$ and $\oplus$, respectively.

With the above in mind, we get

$$\log_{\cdot}(1+A) = \sum_{n \geq 1} \frac{(-1)^{n-1} A^{\cdot n}}{n}, \qquad \exp_{\oplus}(A) = \sum_{n \geq 0} \frac{A^{\oplus n}}{n!}.$$

Then, one can show that setting

$$u_B := \bigoplus_{b \in B} X_b,$$

the $q$-partial cumulant $\kappa_{[r]}^{(q)}(\boldsymbol{u})$ evaluated at $q = 0$ coincides with the conditional cumulant $\kappa(X_1, \dots, X_r)$ up to a sign:

$$\kappa_{[r]}^{(0)}(\boldsymbol{u}) = (-1)^{r-1} \kappa(X_1, \dots, X_r).$$

### 3.1.2 Graph cumulants

Although the cumulants originate from the probability theory, the $q$-deformation introduced here is also relevant in the context of certain graph invariants, called *inversion polynomials*. Let $G = (V, E)$

be a multigraph (i.e. a graph with multiple loops and multiple edges allowed) and for any subset of vertices $I \subseteq V$, we denote by $e_I$ the number of edges in $G$ connecting vertices in $I$. It was shown in [14] that for the family $\boldsymbol{u}$ defined by

$$u_I := q^{e_I},$$

the asociated $q$-partial cumulant $\kappa^q_V(\boldsymbol{u})$ is equal to the $G$-inversion polynomial $\mathcal{I}_G(q)$ (which is also equal to the evaluation of the Tutte polynomial $\mathrm{Tutte}_G(1,q)$ and to the generating series of $G$-parking function; the fact that will not be used in this paper). In particular, it is a polynomial in $q$ with non-negative integer coefficients and it was used to prove positivity results for the $q$-partial cumulants of Macdonald polynomials (we postpone its precise definition to Section 3.5.1, where we use it to provide certain explicit combinatorial formulae). In the following, we show another positivity property of cumulants constructed using multigraphs. This positivity property will be crucial for our first applications.

Suppose that $\boldsymbol{u}$ is a family as in Definition 3.1 and let $G$ be a multigraph with the vertex set $V$. Define the family $\boldsymbol{u}^G$ by setting

$$u^G_I := q^{e_I} u_I$$

for any subset $I \subseteq V$. Finally, for any set partition $\pi \in \mathcal{P}(I)$, define a family $\boldsymbol{u}(\pi) := (\tilde{u}_B)_{B \subseteq \pi}$ by setting $\tilde{u}_B := u_{\cup B}$ (note that for $B \subseteq \pi \in \mathcal{P}(I)$, one has $\bigcup B \subseteq I$, so that $\boldsymbol{u}(\pi)$ is well defined).

**Theorem 3.1.** The $q$-partial cumulant $\kappa^{(q)}_I(\boldsymbol{u}^G)$ is a $q$-positive combination of the $q$-partial cumulants $\kappa^{(q)}_\pi(\boldsymbol{u}(\pi))$, where $\pi \in \mathcal{P}(I)$.

*Proof.* We will prove this by induction on $|I|$. For $|I| = 1$, the statement is obvious. To prepare for the inductive step, we separate the summand corresponding to $\{\{i\} \mid i \in I\} \in \mathcal{P}(I)$ in (3.1). Additionally, we note that for every $\pi \in \mathcal{P}(I)$, the exponent of $q$ in the corresponding summand is greater or equal to the number of loops in $G$. This gives

$$\kappa^{(q)}_I(\boldsymbol{u}^G) = q^{\sum_{i \in I} e_i}\left(\kappa^{(q)}_I(\boldsymbol{u}) + (q-1)^{2-|I|} \sum_{\substack{\pi \in \mathcal{P}(I),\\ |\pi|<|I|}} (-1)^{|\pi|-1}(|\pi|-1)! \left[\sum_{B \in \pi} \tilde{e}_B\right]_q \prod_{B \in \pi} u_B\right),$$

where $e_i := e_{\{i\}}$, $\tilde{e}_B$ denotes the number of edges connecting distinct vertices in $B$ (in particular $\tilde{e}_i := \tilde{e}_{\{i\}} = 0$ for all $i \in I$), and $[n]_q := \frac{q^n-1}{q-1} = \sum_{i=0}^{n-1} q^i$ is the standard numerical factor.

Let $G'$ be the graph $G$ restricted to the vertices from $I$ and with all the loops removed. We arbitrarily order edges of $G'$. Note that the number $e(G')$ of edges of $G'$ is equal to $\sum_{\substack{B \subseteq I,\\ |B|=2}} \tilde{e}_B$ and $e_B(G') = \tilde{e}_B$ for any subset $B \subseteq I$. We are going to construct graphs $G_1, \dots, G_{e(G')}$ such that

$$(q-1)^{2-|I|} \sum_{\substack{\pi \in \mathcal{P}(I),\\ |\pi|<|I|}} (-1)^{|\pi|-1}(|\pi|-1)! \left[\sum_{B \in \pi} e_B(G')\right]_q \prod_{B \in \pi} u_B = \sum_{k=1}^{e(G')} \kappa^{(q)}_{\pi_k}(\boldsymbol{u}(\pi_k)^{G_k}). \qquad (3.3)$$

Let $1 \leq i \leq e(G')$, denote by $E_i(G')$ the set of the first $i$ edges in $G'$, and let $\{m, n\}$ be the set of endpoints of the $i$-th edge in $G'$. Define the graph $G_i$ as follows: its set of vertices is equal to the elements of the set partition $\sigma_i \in \mathcal{P}(I)$ of size $|\sigma_i| = |I| - 1$ (here, $|\sigma_i|$ denotes the number of parts into which $\sigma_i$ divides $I$), whose unique element of size 2 is equal to $\{m, n\}$. For each pair $\{\{k\}, \{l\}\} \in \sigma_i \setminus \{\{m, n\}\}$, the number of edges linking vertices $\{k\}$ and $\{l\}$ in $G_i$ is given by the number of edges in $E_i(G')$ with endpoints $\{k, l\}$. For each $k \in \pi_i \setminus \{\{m, n\}\}$, the number of edges linking vertices $\{k\}$ and $\{\{m, n\}\}$ in $G_i$ is given by the number of edges in $E_i(G')$ with endpoints $\{k, m\}$ or $\{k, n\}$. Finally, the number of loops attached to the vertex $\{\{m, n\}\}$ is given by the number of edges in $E_{i-1}(G')$ with endpoints $\{m, n\}$.

Let us prove by induction on $e(G')$ that the constructed graphs satisfy Eq. (3.3). Clearly, when $G'$ has no edges, both hand sides of Eq. (3.3) are equal to 0. Suppose that $e(G') > 0$ and let $G''$ denote the graph obtained from $G'$ by removing its largest edge $\{m, n\}$. Then

$$
\begin{aligned}
&(q-1)^{2-|I|} \sum_{\substack{\pi \in \mathcal{P}(I),\\ |\pi| < |I|}} (-1)^{|\pi|-1}(|\pi|-1)! \left[\sum_{B\in\pi} e_B(G')\right]_q \prod_{B\in\pi} u_B = \\
&(q-1)^{2-|I|} \sum_{\substack{\pi \in \mathcal{P}(I),\\ |\pi| < |I|}} (-1)^{|\pi|-1}(|\pi|-1)! \left[\sum_{B\in\pi} e_B(G'')\right]_q \prod_{B\in\pi} u_B + \\
&(q-1)^{2-|I|} \sum_{\pi \in \mathcal{P}(\pi_{e(G')})} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{B\in\pi} q^{e_B(G'')} u_{\cup B}.
\end{aligned}
$$

By the inductive hypothesis, we have

$$
(q-1)^{2-|I|} \sum_{\substack{\pi \in \mathcal{P}(I),\\ |\pi| < |I|}} (-1)^{|\pi|-1}(|\pi|-1)! \left[\sum_{B\in\pi} e_B(G'')\right]_q \prod_{B\in\pi} u_B = \sum_{k=1}^{e(G'')} \kappa^{(q)}_{\pi_k}(\boldsymbol{u}(\pi_k)^{G_k}).
$$

Moreover, strictly from the construction of $G_{e(G')}$, we have that $e_B(G_{e(G')}) = e_B(G') - 1 = e_B(G'')$ for any $\{\{m, n\}\} \subset B \subset \pi_{e(G')}$ and $e_B(G_{e(G')}) = e_B(G') = e_B(G'')$ for any $B \subset \pi_{e(G')} \setminus \{\{m, n\}\}$. Therefore,

$$
(q-1)^{2-|I|} \sum_{\pi \in \mathcal{P}(\pi_{e(G')})} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{B\in\pi} q^{e_B(G'')} u_{\cup B} = \kappa^{(q)}_{\pi_{e'(G)}}(\boldsymbol{u}(\pi_{e'(G)})^{G_{e'(G)}}),
$$

which finishes the proof. □

## 3.2 LLT cumulants

Recall from Section 2.2 that we have different normalizations of LLT polynomials. Similarly, we can define different LLT cumulants that follow the normalizations of their polynomial counterparts.

### 3.2.1 LLT cumulants with respect to different normalizations

Let $\boldsymbol{\lambda}/\boldsymbol{\mu} = (\lambda^1/\mu^1, \ldots, \lambda^m/\mu^m)$ be an $m$-tuple of skew Young diagrams. For any surjective function $f : [m] \to [r]$, we say that a pair $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$ is an *r-colored tuple* of skew Young diagrams and we will think of it as an $m$-tuple colored by $r$ colors, so that the $i$-th element $\lambda^i/\mu^i$ has color $f(i)$. For an *$r$-colored tuple* of skew Young diagrams $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$ and for a subset $B \subseteq [r]$, we define $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B$ to be the sub-tuple of $\boldsymbol{\lambda}/\boldsymbol{\mu}$ colored by the colors from $B$. More formally, $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B := (\lambda^{i_1}/\mu^{i_1}, \dots, \lambda^{i_k}/\mu^{i_k})$, where $f^{-1}(B) = \{i_1, \dots, i_k\}$ and $i_1 < \cdots < i_k$.

For a given $r$-colored tuple of skew Young diagrams $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$, we define the *LLT cumulants* (with respect to different normalizations) by the following formulae:

$$\kappa_{\mathrm{LLT}^{\mathrm{cospin}}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f) := \kappa^{(q)}_{[r]}(\boldsymbol{u}(\mathrm{LLT}^{\mathrm{cospin}})), \text{ where } u(\mathrm{LLT}^{\mathrm{cospin}})_B := \mathrm{LLT}^{\mathrm{cospin}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B, \tag{3.4}$$

$$\kappa_{\mathrm{LLT}^{\mathrm{Mac}}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f) := \kappa^{(q)}_{[r]}(\boldsymbol{u}(\mathrm{LLT}^{\mathrm{Mac}})), \text{ where } u(\mathrm{LLT}^{\mathrm{Mac}})_B := \mathrm{LLT}^{\mathrm{Mac}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B, \tag{3.5}$$

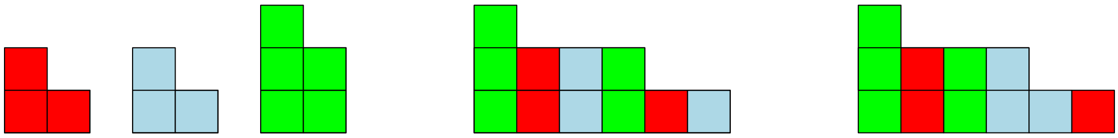
$$\kappa_{\mathrm{LLT}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f) := \kappa^{(q)}_{[r]}(\boldsymbol{u}(\mathrm{LLT})), \text{ where } u(\mathrm{LLT})_B := \mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B. \tag{3.6}$$

Note that for any $m$-tuple of skew Young diagrams $\boldsymbol{\lambda}/\boldsymbol{\mu}$, there exists the unique 1-colored tuple of skew Young diagrams $(\boldsymbol{\lambda}/\boldsymbol{\mu}, \pi^{[m]}_{[1]})$, where $\pi^{[m]}_{[1]}$ is the unique surjection of $[m]$ onto $\{1\}$. In this case, the cumulants $\kappa_{\mathrm{LLT}^{\mathrm{cospin}}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, \mathrm{id}_{[m]})$, $\kappa_{\mathrm{LLT}^{\mathrm{Mac}}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, \mathrm{id}_{[m]})$, and $\kappa_{\mathrm{LLT}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, \mathrm{id}_{[m]})$ coincide with the associated LLT functions $\mathrm{LLT}^{\mathrm{cospin}}(\boldsymbol{\lambda}/\boldsymbol{\mu})$, $\mathrm{LLT}^{\mathrm{Mac}}(\boldsymbol{\lambda}/\boldsymbol{\mu})$, and $\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu})$, respectively. In general, LLT-cumulants can be interpreted as an $r$-colored generalization of LLT polynomials.

### 3.2.2 Cumulants of symmetric functions

The concept of $r$-colored tuples of skew shapes arose from the definition of cumulants of symmetric functions naturally indexed by partitions. This definition was introduced in [15] (in the context of Jack and Macdonald symmetric functions) as follows: let $(f_\lambda)_\lambda$ be a class of symmetric functions indexed by partitions. For partitions $\lambda^1, \dots, \lambda^r$, we define the family $(\boldsymbol{u})$ indexed by subsets of $[r]$ as $u_B := f_{\lambda^B}$, where the partition $\lambda^B := \bigoplus_{i \in B} \lambda^i$ is obtained from partitions $\lambda^i$, $i \in B$ by summing their coordinates: $\lambda^B_j := \sum_{i \in B} \lambda^i_j$. We observe that the data of partitions $\lambda^1, \dots, \lambda^r$ can be alternatively encoded as an $r$-colored partition $(\lambda = \lambda^{[r]}, f)$ as follows: let $\lambda$ be a partition and let $f\colon [\ell(\lambda)] \to [r]$ be a surjective function (that we interpret as the coloring of columns of the Young diagram $\lambda$ by $r$ colors), such that the Young diagram formed by columns colored by $i$ is equal to $\lambda^i$. Then, it is clear that for every $B \subseteq [r]$, the Young diagram formed by columns colored by colors in $B$ is equal to $\lambda^B$.

Of course, there are many colorings $f\colon [\ell(\lambda)] \to [r]$ which encode partitions $\lambda^1, \dots, \lambda^r$ as an $r$-colored partition $(\lambda, f)$, but among them there is a canonical choice, which we call the *canonical coloring* $(\lambda, f_{\mathrm{cc}}\colon [\ell(\lambda)] \to [r])$. It is uniquely determined by the following property: for any $i < j$ such that $\lambda^t_i = \lambda^t_j$ (here $\lambda^t$ denotes the *transpose* of $\lambda$, i.e., the diagram with $\lambda_1$ boxes in the first column, $\lambda_2$ boxes in the second column, etc.), one has $f_{\mathrm{cc}}(i) \leq f_{\mathrm{cc}}(j)$. This property simply means that the Young diagram $\lambda^{[r]}$ can be obtained by sorting the columns of $\lambda^1, \dots, \lambda^r$ such that all the columns of the same height are ordered with respect to the natural order $1 < \cdots < r$, see Fig. 3.1.



$\lambda^1$ $\lambda^2$ $\lambda^3$ canonically colored Young diagram associated with $\lambda^1, \lambda^2, \lambda^3$ $r$-colored Young diagram associated with $\lambda^1, \lambda^2, \lambda^3$

Figure 3.1: $r$-colored and canonically colored partitions.

## 3.3 Macdonald cumulants

When $f_\lambda = \tilde{H}_\lambda^{(q,t)}$ is the transformed version of the Macdonald polynomial indexed by a partition $\lambda$, the corresponding $q$-partial cumulant $k_{[r]}^{(q)}(\boldsymbol{u})$ is called the *Macdonald cumulant* and denoted $\kappa(\lambda^1, \dots, \lambda^r)(x; q, t)$:

$$\kappa(\lambda^1, \dots, \lambda^r)(x; q, t) := k_{[r]}^{(q)}(\boldsymbol{u}). \tag{3.7}$$

Macdonald cumulants were studied in [13, 14], where their polynomiality and combinatorial interpretation was obtained, generalizing the celebrated HHL formula (2.6). Furthermore, it was conjectured in [13] that Macdonald cumulants are Schur-positive:

**Conjecture 2** ([13]). *Let $\lambda^1, \dots, \lambda^r$ be partitions. Then for any partition $\nu$ the coefficient $[s_\nu]\kappa(\lambda^1, \dots, \lambda^r)$ in the Schur expansion of the Macdonald cumulant is a polynomial in $q, t$ with nonnegative coefficients.*

### 3.3.1 Schur-positivity conjecture for LLT cumulants

The first motivation for introducing LLT cumulants is to attack Conjecture 2. Since Macdonald polynomials can be naturally decomposed into LLT polynomials, it is natural to ask whether a similar decomposition occurs for Macdonald cumulants. Moreover, it was proved by Grojnowski and Haiman [25] that LLT polynomials are Schur-positive, which gives an alternative proof of the Schur positivity of Macdonald polynomials. Extensive computer simulations performed by the authors tend us to believe that the result of Grojnowski and Haiman might be a special case of Schur-positivity that holds for the more general class of LLT cumulants. Therefore, we propose the following conjecture:

**Conjecture 3.** *For any $r$-colored tuple of skew shapes $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$ and for any partition $\nu$ the coefficient $[s_\nu]\kappa_{\mathrm{LLT}^{\mathrm{cospin}}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$ in the Schur expansion of the LLT cumulant is a polynomial in $q$ with nonnegative integer coefficients.*

**Example.** *As an example of Conjecture 3, we have*

$$\begin{aligned}\kappa((2,2)/(1),(2),(1,1)) = {} & (q^6 + 2q^5 + 3q^4 + 4q^3 + 4q^2 + 4q + 2)s_{(2,2,1,1,1)} \\ & + (q^5 + 2q^4 + 3q^3 + 4q^2 + 4q + 2)s_{(2,2,2,1)} \\ & + (q^6 + 2q^5 + 3q^4 + 4q^3 + 4q^2 + 4q + 2)s_{(3,1,1,1,1)}\end{aligned}$$

$$
\begin{aligned}
&+ (2q^5 + 5q^4 + 8q^3 + 10q^2 + 10q + 6)s_{(3,2,1,1)} \\
&+ (2q^4 + 4q^3 + 6q^2 + 7q + 4)s_{(3,2,2)} \\
&+ (q^4 + 3q^3 + 5q^2 + 6q + 4)s_{(3,3,1)} \\
&+ (q^5 + 3q^4 + 5q^3 + 6q^2 + 6q + 4)s_{(4,1,1,1)} \\
&+ (q^4 + 4q^3 + 7q^2 + 8q + 6)s_{(4,2,1)} + (q^2 + 2q + 2)s_{(4,3)} \\
&+ (q^3 + 2q^2 + 2q + 2)s_{(5,1,1)} + (q^2 + 2q + 2)s_{(5,2)}.
\end{aligned}
$$

### 3.3.2 The relation between Schur-positivity of LLT cumulants and Macdonald cumulants

In the following, we prove that Conjecture 3 implies Conjecture 2. In order to do this, we express Macdonald cumulants as a positive linear combination of LLT cumulants, generalizing the classical decomposition from Theorem 2.2 to cumulants, and we show that Schur-positivity of LLT cumulants can be put into the following hierarchy: Schur-positivity of $\kappa_{\mathrm{LLT}^{\mathrm{cospin}}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$ implies Schur-positivity of $\kappa_{\mathrm{LLT}^{\mathrm{Mac}}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$, which further implies Schur-positivity of $\kappa_{\mathrm{LLT}^{\mathrm{inv}}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$.

**Remark 2.** *In fact, the chain of implications mentioned above is valid only when $\boldsymbol{\lambda}/\boldsymbol{\mu}$ is a sequence of ribbon shapes due to the definition of the normalization $\mathrm{LLT}^{\mathrm{Mac}}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ (see (2.5)). However, Schur-positivity of $\kappa_{\mathrm{LLT}^{\mathrm{cospin}}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$ implies Schur-positivity of $\kappa_{\mathrm{LLT}^{\mathrm{inv}}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$ for all sequences $\boldsymbol{\lambda}/\boldsymbol{\mu}$.*

**Theorem 3.2.** Let $\nu^1, \dots, \nu^r$ be partitions. Then, the following identity holds true:

$$
\kappa(\nu^1, \dots, \nu^r)(x; q, t) = \sum_{\boldsymbol{\lambda}/\boldsymbol{\mu}} t^{\mathrm{maj}(\boldsymbol{\lambda}/\boldsymbol{\mu})} \kappa_{\mathrm{LLT}^{\mathrm{Mac}}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f_{\mathrm{cc}}), \tag{3.8}
$$

where we sum over all tuples of ribbons whose bottom, far-right cell has content $0$ and such that $|\boldsymbol{\lambda}_i/\boldsymbol{\mu}_i| = (\nu^{[r]})^t_j$ for $1 \leq i \leq \ell(\nu^{[r]})$ (i.e. the size of the $i$-th ribbon is equal to the length of the $i$-th column of $\nu^{[r]}$) and $f_{\mathrm{cc}}$ is the canonical coloring associated with $\nu^1, \dots, \nu^r$.

*Proof.* It is a direct consequence of the interpretation of the Macdonald cumulant as the $q$-partial cumulant of the canonically $r$-colored partition and of Theorem 2.2. Indeed, note that for any subset $B \subseteq [r]$, Theorem 2.2 applied to $\nu = \nu^B$ gives

$$
\tilde{H}^{(q,t)}_{\nu^B} = \sum_{\boldsymbol{\lambda}/\boldsymbol{\mu}} t^{\mathrm{maj}(\boldsymbol{\lambda}/\boldsymbol{\mu})}\, \mathrm{LLT}^{\mathrm{Mac}}(\boldsymbol{\lambda}/\boldsymbol{\mu}),
$$

where we sum over skew-partitions whose $i$-th element is a ribbon of length $(\nu^B)^t_j$ whose bottom, far-right cell has content $0$. In particular, for any set-partition $\pi \in \mathcal{P}([r])$, one has

$$
\prod_{B\in\pi} \tilde{H}^{(q,t)}_{\nu^B} = \sum_{\boldsymbol{\lambda}/\boldsymbol{\mu}\in\mathcal{A}} \prod_{B\in\pi} t^{\mathrm{maj}((\boldsymbol{\lambda}/\boldsymbol{\mu}, f_{\mathrm{cc}})^B)}\, \mathrm{LLT}^{\mathrm{Mac}}((\boldsymbol{\lambda}/\boldsymbol{\mu}, f_{\mathrm{cc}})^B),
$$

where the sum runs over the same set as the summation in (3.8).

Therefore, for any set-partition $\pi \in \mathcal{P}([r])$, one has $\sum_{B\in\pi} \mathrm{maj}((\boldsymbol{\lambda}/\boldsymbol{\mu}, f_{\mathrm{cc}})^B) = \mathrm{maj}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ and the formula (3.8) follows. ☐

**Theorem 3.3.** Suppose that Conjecture 3 holds true. Then, for any $r$-colored tuple of skew shapes $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$ and for any partition $\nu$, the coefficients

$$[s_\nu]\kappa_{\mathrm{LLT}^{\mathrm{Mac}}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f) \in \mathbb{Z}_{\geq 0}[q], \qquad [s_\nu]\kappa_{\mathrm{LLT}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f) \in \mathbb{Z}_{\geq 0}[q]$$

are polynomials in $q$ with nonnegative integer coefficients. In particular, Conjecture 2 holds true.

*Proof.* Recall the definiton Eq. (3.4) of LLT cumulants. We will show that there exist graphs $G \subseteq G'$ such that

$$\kappa^{(q)}_{[r]}(\boldsymbol{u}(\mathrm{LLT})) = \kappa^{(q)}_{[r]}(\boldsymbol{u}(\mathrm{LLT}^{\mathrm{cospin}})^{G'}), \tag{3.9}$$

$$\kappa^{(q)}_{[r]}(\boldsymbol{u}(\mathrm{LLT}^{\mathrm{Mac}})) = \kappa^{(q)}_{[r]}(\boldsymbol{u}(\mathrm{LLT}^{\mathrm{cospin}})^{G}). \tag{3.10}$$

Then the statements follow directly from Theorem 3.1 and Eq. (3.8).

Notice that a family of nonnegative integers $(e_B)_{B\subseteq V}$ indexed by subsets of the set $V$ corresponds to the number of edges in some graph $G = (V, E)$ linking vertices in $B$ if and only if

$$e_B \geq \sum_{b\in B} e_{\{b\}} \quad \text{and} \quad e_B = \sum_{\substack{B'\subset B,\\ |B'|=2}} e_{B'} - (|B| - 2)\sum_{b\in B} e_{\{b\}}. \tag{3.11}$$

Indeed, the inequality corresponds to $e_B$ counting all the loops on vertices from $B$, and the equality counts the edges between each pair of vertices from $B$ minus the overcounted loops.

To show (3.9), consider $e_B = \min_{T\in\mathrm{SSYT}((\boldsymbol{\lambda}/\boldsymbol{\mu},f)^B)} \mathrm{inv}(T) = |\,\mathrm{Inv}(T) \setminus \mathrm{Inv}_{\mathrm{cospin}}(T)|$, which does not depend on the choice of $T \in \mathrm{SSYT}((\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B)$. Then the conditions in (3.11) are satisfied since for a pair of boxes $\square \in (\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^{\{i\}}$ and $\square' \in (\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^{\{j\}}$, one has $(\square, \square') \in \mathrm{Inv}(T) \setminus \mathrm{Inv}_{\mathrm{cospin}}(T)$ if and only if $(\square, \square') \in \mathrm{Inv}(T_{\{i,j\}}) \setminus \mathrm{Inv}_{\mathrm{cospin}}(T_{\{i,j\}})$, where $T_{\{i,j\}}$ is a tableau $T$ restricted to $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^{\{i,j\}}$.

Similarly, for

$$e_B = a((\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B) = \sum_{\square\in\mathrm{Des}((\boldsymbol{\lambda}/\boldsymbol{\mu},f)^B)} |\{\square' \mid c(\square') = c(\square), \tilde{c}(\square') > \tilde{c}(\square)\}|,$$

one has

$$\sum_{\square\in\mathrm{Des}((\boldsymbol{\lambda}/\boldsymbol{\mu},f)^B)} |\{\square' \mid c(\square') = c(\square), \tilde{c}(\square') > \tilde{c}(\square)\}| = \tag{3.12}$$

$$\sum_{b\in B}\ \sum_{\square\in\mathrm{Des}((\boldsymbol{\lambda}/\boldsymbol{\mu},f)^{\{b\}})}\ \sum_{b'\in B} |\{\square' \in (\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^{\{b'\}} \mid c(\square') = c(\square), \tilde{c}(\square') > \tilde{c}(\square)\}|. \tag{3.13}$$

Note that for any subset $A \subseteq B$ and for each pair of boxes $(\square, \square') \in (\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^A$, there is a uniquely associated pair of boxes $(\square, \square') \in (\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B$ and their contents are identical in $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^A$ and $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B$, while their shifted contents might be different but the relation $\tilde{c}(\square') > \tilde{c}(\square)$ is again the same in both $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^A$ and $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B$. This observation together with (3.12) implies that the quantities $e_B$ satisfy (3.11). This proves (3.10).

Finally, let us prove that $a((\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B) \leq \min_{T\in\mathrm{SSYT}((\boldsymbol{\lambda}/\boldsymbol{\mu},f)^B)} \mathrm{inv}(T)$. Let $\square \in \mathrm{Des}((\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B)$ and $\square' \in (\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B$ be such that $c(\square') = c(\square)$ and $\tilde{c}(\square') > \tilde{c}(\square)$. For any $T \in \mathrm{SSYT}((\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B)$, we necessarily have $T(\square) > T(\square_\downarrow)$. Therefore, either $(\square, \square') \in \mathrm{Inv}(T)$ or $(\square', \square_\downarrow) \in \mathrm{Inv}(T)$ (or both). Summing over all $\square \in \mathrm{Des}((\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B)$ and

$$\square' \in \{\square' \in (\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B \mid c(\square') = c(\square), \tilde{c}(\square') > \tilde{c}(\square)\},$$

we have that $a((\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B) \leq \mathrm{inv}(T)$ for any $T \in \mathrm{SSYT}((\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B)$. Thus, $a((\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B) \leq \min_{T\in\mathrm{SSYT}((\boldsymbol{\lambda}/\boldsymbol{\mu},f)^B)} \mathrm{inv}(T)$, which means that $[s_\nu]\kappa_{\mathrm{LLT}^{\mathrm{Mac}}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$ and $[s_\nu]\kappa_{\mathrm{LLT}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$ are indeed polynomials in $q$, which finishes the proof. $\square$

## 3.4 Graph colorings

In the following, we interpret LLT polynomials as the generating functions of colorings of certain directed graphs. This viewpoint provides a natural graph-theoretic interpretation of LLT cumulants as well as various positivity properties generalizing some recent results [5, 14].

### 3.4.1 LLT graphs

**Definition 3.3.** We call $G$ an *LLT graph* if it is a finite directed graph with three types of edges, visually depicted as $\rightarrow$, $\twoheadrightarrow$, and $\Rightarrow$, which we call *edges of type I*, *of type II*, and *double edges*, respectively. Denote the corresponding sets of edges by $E_1(G)$, $E_2(G)$, and $E_d(G)$. Additionally, write $\mathscr{G}$ for the $\mathbb{Z}[q]$-module spanned by LLT graphs and $\mathscr{G}_1 < \mathscr{G}$ for the submodule generated by LLT graphs with only edges of type II ($E_1(G) = E_d(G) = \emptyset$).

Let $\mathrm{QSym}$ denote the ring of quasi-symmetric functions over $\mathbb{Z}[q]$: a family formally introduced in [21]. We recall that a quasisymmetric function $f$ is a power series in variables $x_1, x_2, \ldots$ of a bounded degree such that for any sequence of positive integers $(\alpha_1, \ldots, \alpha_n)$ the coefficients of the monomial $[x_{i_1}^{\alpha_1} \cdots x_{i_n}^{\alpha_n}]f$ in $f$ is the same for all possible choices of indices $i_1 < \cdots < i_n$. With an LLT graph $G$ we associate its *LLT polynomial*:

$$\mathrm{LLT}(G) := \sum_{f:V(G)\to\mathbb{Z}_{>0}} \left(\prod_{(u,v)\in E(G)} \varphi_f(u,v)\right) \cdot \left(\prod_{v\in V(G)} x_{f(v)}\right), \tag{3.14}$$

with

$$\varphi_f(u,v) = \begin{cases} [f(u) > f(v)] & \text{for } (u,v) \in E_1(G); \\ [f(u) \geq f(v)] & \text{for } (u,v) \in E_2(G); \\ q[f(u) > f(v)] + [f(u) \leq f(v)] & \text{for } (u,v) \in E_d(G), \end{cases} \tag{3.15}$$

where $[A]$ is the characteristic function of condition $A$, i.e., is equal to 1 if $A$ is true and 0 otherwise.

There is an obvious way to associate an LLT graph $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ to a sequence of skew shapes $\boldsymbol{\lambda}/\boldsymbol{\mu}$ such that $\mathrm{LLT}(G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}) = \mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu})$. To be precise, vertices correspond to cells, edges of type I go from a

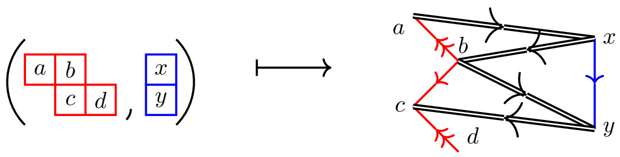



Figure 3.2: The LLT graph corresponding to ((3,2)/(1), (1,1)).

cell $\square$ to $\square_{\downarrow}$, edges of type II go from a cell $\square$ to $\square_{\leftarrow}$, and double edges connect cells that correspond to inversions (see Section 3.4.1).

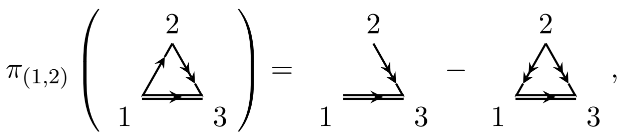
### 3.4.2 Local transformations on LLT graphs

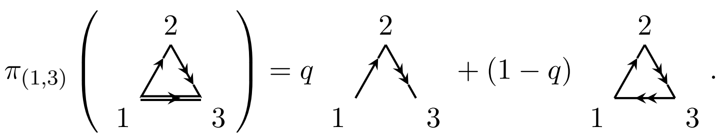
Let $G$ be an LLT graph and let $\vec{e_i} \in E_i(G)$ for $i \in \{1, d\}$. Define the local transformation

$$\pi_{\vec{e_i}}(G) = \begin{cases} G \setminus \{\vec{e_1}\} - G_{\vec{e_1} \to \overleftarrow{e_2}} & \text{for } i = 1, \\ qG \setminus \{\vec{e_d}\} + (1-q) G_{\vec{e_d} \to \overleftarrow{e_2}} & \text{for } i = d, \end{cases}$$

where $\vec{e_i} \to \overleftarrow{e_j}$ ($\vec{e_i} \to \vec{e_j}$, respectively) denotes replacing the directed edge $\vec{e_i}$ of type $i$ by the edge of type $j$ with the opposite (the same, respectively) direction.

**Example.** *We have*




We define

$$\pi(G) := \Bigg( \prod_{\vec{e} \in E_1(G) \cup E_d(G)} \pi_{\vec{e}} \Bigg)(G)$$

as the concatenation of local transformations over all edges of type I and double edges (these transformations are commutative so their order does not matter and this concatenation is well-defined). Note that local transformations kill all edges of type I and double edges and thus, the map $\pi\colon \mathscr{G} \to \mathscr{G}_1$ is well-defined. In fact, we claim that the map $\mathrm{LLT}\colon \mathscr{G} \to \mathrm{QSym}$ is a well-defined surjective homomorphism such that $\mathrm{LLT}(G) = \mathrm{LLT} \circ \pi(G)$ for every LLT graph.

**Lemma 3.4.** For $\mathscr{G}$ and $\mathscr{G}_1$ as in Definition 3.3, the following diagram is commutative:

$$\begin{tikzcd} \mathscr{G} \arrow[r, "\pi"] \arrow[dr, "\mathrm{LLT}"'] & \mathscr{G}_1 \arrow[d, "\mathrm{LLT}"] \\ & \mathrm{QSym} \end{tikzcd}$$

*Proof.* Let $G$ be an LLT graph. It was proved by Féray [17] that $\mathrm{LLT}\colon \mathscr{G}_1 \to \mathrm{QSym}$ is a well-defined surjective homomorphism. Moreover, it is straightforward from the definition of the map LLT that it is invariant under the local transformations, i.e., for every $\vec{e} \in E_1(G) \cup E_d(G)$, one has $\mathrm{LLT}(\pi_{\vec{e}}(G)) = \mathrm{LLT}(G)$. Thus, $\mathrm{LLT}(G) = \mathrm{LLT}(\pi(G))$, which finishes the proof. $\square$

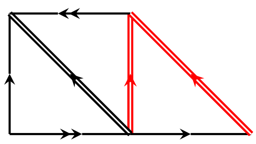
**Remark 3.** *Let $\widehat{\mathscr{G}_1} < \mathscr{G}_1$ be a submodule of $\mathscr{G}_1$ spanned by acyclic graphs. The main result of Féray [17] is an explicit description of the kernel of the map $\mathrm{LLT}\colon \widehat{\mathscr{G}_1} \to \mathrm{QSym}$ by using the* cyclic inclusion-exclusion principle*. This description together with Lemma 3.4 can be, a priori, used to describe the kernel of the morphism $\mathrm{LLT}\colon \mathscr{G} \to \mathrm{QSym}$ and thus to understand all the relations between LLT graphs under the* LLT *morphism. Additionally, $\mathscr{G}$ seems to carry a natural Hopf algebra structure.*

*Studying various relations between LLT polynomials is a very active topic recently and it proved to be useful in understanding the combinatorial structure of LLT polynomials [47, 32, 2, 5, 18]. We believe that further studies in the direction of understanding the algebraic structure of the pair $(\mathscr{G}, \mathrm{LLT})$ might bring better understanding of the combinatorial structure of LLT polynomials, and we leave this problem for future research.*

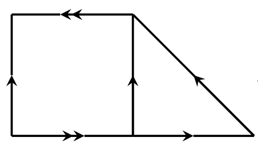
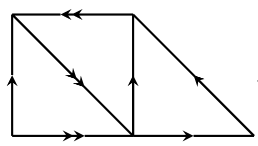
As a consequence of Lemma 3.4 and its proof, we obtain two identities expressing the LLT polynomial of a given LLT graph $G$ in terms of two important LLT graphs, which do not have any double edges. For any subset $E \subseteq E_d(G)$, we define $G^E$ and $\tilde{G}^E$ as follows:

- $V(\tilde{G}^E) = V(G^E) = V(G)$,
- $E_d(\tilde{G}^E) = E_d(G^E) = \emptyset$,
- $E_1(\tilde{G}^E) = E_1(G^E) = E_1(G) \cup E$,
- $E_2(\tilde{G}^E) = E_2(G) \cup \{(u,v) \mid (v,u) \in E_d \setminus E\}$, and $E_2(G^E) = E_2(G)$.

**Example.** *For*

$G =$ 

*and $E \subseteq E_d(G)$ equal to the set of the red edges above, we have*

$G^E =$  ,  $\tilde{G}^E =$  .

**Corollary 1.** *For any LLT graph $G$, we have*

$$\mathrm{LLT}(G) = \sum_{E \subseteq E_d(G)} q^{|E|}\, \mathrm{LLT}(\tilde{G}^E) = \sum_{E \subseteq E_d(G)} (q-1)^{|E|}\, \mathrm{LLT}(G^E).$$

*Proof.* Note that

$$\Big(\prod_{\vec{e}\in E_d(G)} \pi'_{\vec{e}}\Big)(G) = \sum_{E\subseteq E_d(G)} q^{|E|}\tilde{G}^E, \quad \Big(\prod_{\vec{e}\in E_d(G)} \pi''_{\vec{e}}\Big)(G) = \sum_{E\subseteq E_d(G)} (q-1)^{|E|}G^E,$$

where $\pi'_{\vec{e_d}} = G_{\vec{e_d}\to\vec{e_2}} + qG_{\vec{e_d}\to\vec{e_1}}$ and $\pi''_{\vec{e_d}} = G\setminus\{\vec{e_d}\} + (q-1)G_{\vec{e_d}\to\vec{e_1}}$ for $e_d\in E_d(G)$. Moreover, $\mathrm{LLT}(G) = \mathrm{LLT}(\pi'_{\vec{e_d}})$ follows from the definition, and $\mathrm{LLT}(G) = \mathrm{LLT}(\pi''_{\vec{e_d}})$ follows from

$$\mathrm{LLT}(G) = \mathrm{LLT}(G_{\vec{e_d}\to\vec{e_2}}) + q\,\mathrm{LLT}(G_{\vec{e_d}\to\vec{e_1}}) = \mathrm{LLT}(G\setminus\{\vec{e_d}\}) + (q-1)\,\mathrm{LLT}(G_{\vec{e_d}\to\vec{e_1}})$$

since we have $\mathrm{LLT}(G_{\vec{e_d}\to\vec{e_2}}) = \mathrm{LLT}(G\setminus\{\vec{e_d}\}) - \mathrm{LLT}(G_{\vec{e_d}\to\vec{e_1}})$. □

**Remark 4.** *Observe that the first equation in Corollary 1 is, in fact, a special case of a more general formula: for any set-partition $\pi\in\mathcal{P}([r])$, one has*

$$\prod_{B\in\pi}\mathrm{LLT}(G|_{V_B}) = \sum_{E\subseteq E_d(G)}\mathrm{LLT}(\tilde{G}^E)\prod_{B\in\pi} q^{|E_B|}, \tag{3.16}$$

*where $E_B\subseteq E$ is the subset of edges with both endpoints in $B$. We recall that $G_\pi := \bigoplus_{B\in\pi} G|_{V_B}$ and $\prod_{B\in\pi}\mathrm{LLT}(G|_{V_B}) = \mathrm{LLT}(G_\pi)$ and thus, formula* (3.16) *is proved similarly to Corollary 1.*

### 3.4.3 LLT cumulants of $r$-colored tuples of shapes

The definition of LLT cumulants of $r$-colored tuples of skew-shapes generalizes naturally to the definition of LLT cumulants of $r$-colored LLT graphs.

**Definition 3.4.** We say that $(G,f)$ is an *$r$-colored LLT graph* if $G$ is an LLT graph and $f\in V(G)\to[r]$ is a surjective coloring of vertices of $G$ such that both endpoints of edges in $E_1(G)\cup E_2(G)$ have the same color. For any subset $B\subseteq[r]$, we define the vertex set $V_B := \{v\in V(G)\mid f(v)\in B\}$ and for any subset $V'\subseteq V(G)$, we define $G|_{V'}$ as the subgraph of $G$ obtained by restricting its set of vertices to $V'$. Then, we define the *LLT cumulant* of an $r$-colored LLT graph $(G,f)$ as the $q$-partial cumulant $\kappa^{(q)}_{[r]}(\boldsymbol{u})$ for the family defined by

$$u_B := \mathrm{LLT}(G|_{V_B}).$$

### 3.4.4 Decomposition of the cumulants of $r$-colored LLT graphs

In the following, we prove that the LLT cumulant of the $r$-colored LLT graph $(G,f)$ can be naturally expressed as a sum of LLT polynomials of so-called $f$-connected graphs.

**Definition 3.5.** Let $(G,f)$ be an $r$-colored LLT graph. We say that it is *$f$-connected* if the graph $G_f$ obtained from $G$ by identifying vertices of the same color is connected. In other words, the graph $G$ is $f$-connected if for every pair $i,j\in[r]$, there exists $i=i_0\neq i_1\cdots\neq i_k=j\in[r]$ and vertices $v_0,\dots,v_k\in V(G)$ colored by $i_0,\dots,i_k$ respectively such that $v_{i_{t-1}}$ is connected to $v_{i_t}$ for every $1\le t\le k$.

Note that when $f$ is a bijection, the graph $G$ is $f$-connected if and only if $G$ is connected. Also, if $f$ is a 1-coloring, the condition of being $f$-connected is empty (it is always satisfied). We have the following combinatorial interpretation of an LLT cumulant of an $r$-colored LLT graph $(G, f)$.

**Theorem 3.5.** Let $(G, f)$ be an $r$-colored LLT graph and denote $E_d = E_d(G)$. Then:

$$\kappa_{\mathrm{LLT}}(G, f)(q+1) = \sum_{\substack{E \subseteq E_d \\ G^E\, f\text{-connected}}} q^{|E|-r+1}\, \mathrm{LLT}(G^E)(q+1). \tag{3.17}$$

Theorem 3.5 essentially shows the structure behind the, a priori, algebraic definition of a cumulant: it kills all $f$-disconnected summands in the expansion and preserves the $f$-connected ones. Furthermore, we note that we formulate the statement with the polynomials evaluated at $q+1$ to highlight the LLT-positivity of the cumulant after the shift $q \longmapsto q+1$: an operation that is also connected to the $e$-positivity phenomenon (see Section 3.5.4).

*Proof of Theorem 3.5.* We have

$$\begin{aligned}\kappa_{\mathrm{LLT}}(G, f)(q+1) :&= q^{1-r} \sum_{\pi \in \mathcal{P}([r])} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{B \in \pi} \mathrm{LLT}(G|_{V_B}) \\ &= q^{1-r} \sum_{\pi \in \mathcal{P}([r])} (-1)^{|\pi|-1}(|\pi|-1)!\, \mathrm{LLT}\left(\bigoplus_{B \in \pi} G|_{V_B}\right),\end{aligned}$$

where $G_1 \oplus G_2$ is a disjoint union of the LLT graphs $G_1$ and $G_2$. For simplicity of notation, let us write $G_\pi := \bigoplus_{B \in \pi} G|_{V_B}$.

By Corollary 1, for each $B \in \mathcal{P}([r])$, we get

$$\mathrm{LLT}\left(G_\pi\right)(q+1) = \sum_{E \subseteq E_d(G_\pi)} q^{|E|}\, \mathrm{LLT}(G_\pi^E)(q+1) \tag{3.18}$$

so that

$$\kappa_{\mathrm{LLT}}(G, f)(q+1) = q^{1-r} \sum_{\pi \in \mathcal{P}([r])} (-1)^{|\pi|-1}(|\pi|-1)! \sum_{E \subseteq E_d(G_\pi)} q^{|E|}\, \mathrm{LLT}(G_\pi^E)(q+1). \tag{3.19}$$

Now consider an LLT graph $G' = G_\sigma^E$ for some $\sigma \in \mathcal{P}([r])$ and $E \subseteq E_d(G_\sigma)$. For a fixed $G'$ of this form, pick $\sigma$ minimal, i.e. pick $\sigma$ such that for every block $B \in \sigma$, the graph $G'|_{V_B}$ is $f$-connected. Note that $G'$ is $f$-connected if and only if $\sigma = \{[r]\}$. We compute the contribution of the graph $G'$ to the RHS of the formula (3.19).

Note that $G'$ appears in a summand corresponding to a partition $\pi$ if and only if for every $B \in \sigma$, there exists $C \in \pi$ such that $B \subseteq C$. This is known as the containment relation $\sigma \leq \pi$ on the set of partitions. Therefore, we have

$$[\mathrm{LLT}(G')(q+1)]\kappa_{\mathrm{LLT}}(G, f) = q^{|E|-r+1} \sum_{\sigma \leq \pi} (-1)^{|\pi|-1}(|\pi|-1)! = q^{|E|-r+1} \delta_{\sigma, \{[r]\}}. \tag{3.20}$$

The last equality comes from the well-known fact that $(-1)^{|\pi|-1}(|\pi|-1)!$ is equal to the Möbius function $\mu(\pi, \{[r]\})$ on the poset of set-partitions $(\mathcal{P}([r]), \leq)$ and the sum of the Möbius function $\mu(\pi, \{[r]\})$ over the interval $\pi \in [\sigma, \{[r]\}]$ is non-zero (and equal to 1) only if $\sigma = \{[r]\}$ (see, e.g., [58]). This finishes the proof as $\sigma = \{[r]\}$ if and only if $G'$ is $f$-connected. □

## 3.5 Various positivity results

The purpose of this section is to derive various combinatorial formulae for an LLT cumulant of an $r$-colored LLT graph and prove certain positivity results. We start by a quick review on $G$-inversion polynomials and their different interpretations.

### 3.5.1 $G$-inversion polynomials and Tutte polynomials

Let $G$ be a multigraph (with possible multiedges and multiloops, as previously) on the set of vertices $[r]$. We say that $T$ is a *spanning tree* of $G$ if it is a subgraph of $G$ with the same set of vertices $[r]$ and it is a tree (it is connected and has no cycles). A pair $(i, j)$ is called an *inversion* of a spanning tree $T$ of $G$ if $i, j \neq 1$ and if $i$ is an ancestor of $j$ and $i > j$. An inversion $(i, j)$ is a *$\kappa$-inversion* if, additionally, $j$ is adjacent to the parent of $i$ in $G$. A $G$-inversion polynomial is a generating function of spanning trees of $G$ counted with respect to the number of $\kappa$-inversions.

Let $\tilde{G}$ be a graph obtained from $G$ by replacing all multiple edges by single ones. We recall that for any subset $B \subseteq V$, we denote the number of edges linking vertices in $B$ by $e_B$. The *$G$-inversion polynomial* is given by

$$\mathcal{I}_G(q) = q^{\text{number of loops in } G} \sum_{T \subseteq \tilde{G}} q^{\kappa(T)} \prod_{\{i,j\} \in T} [e_{i,j}(G)]_q, \tag{3.21}$$

where $e_{i,j} := e_{\{i,j\}}$ and the sum runs over all spanning trees of $\tilde{G}$,

$$\kappa(T) = \sum_{\{i,j\}-\kappa-\text{inversion in } T} e_{\text{parent}(i),j}(G), \tag{3.22}$$

and we use the standard notation $[n]_q := \frac{q^n-1}{q-1} = 1 + q + \cdots + q^{n-1}$. As we already mentioned in the introduction, $\mathcal{I}_G(q) = \mathrm{Tutte}(1, q)$, where $\mathrm{Tutte}(x, y)$ is the Tutte polynomial of $G$ (a classical graph invariant introduced by Tutte in [57]):

$$\mathrm{Tutte}_G(x, y) = \sum_{H \subseteq G} (x-1)^{c(H)-1} (y-1)^{|E(H)|-|V|+c(H)}. \tag{3.23}$$

The summation index above runs over all (possibly disconnected) subgraphs of $G$, $c(H)$ denotes the number of connected components of $H$, and $E(H)$ is the set of edges of $H$. In fact, we have the following lemma, which is essentially due to Gessel [22] and Josuat-Vergès [37] (see also [14] for treating both frameworks in the setting of multigraphs).

**Lemma 3.6.** Let $G$ be a multigraph with the vertex set $V = [r]$ and let $\boldsymbol{u}$ be a family indexed by subsets of $[r]$ defined as $u_B := q^{e_B}$ for every $B \subseteq [r]$. Then we have the following equalities between the generating series:

$$\mathcal{I}_G(q) = \mathrm{Tutte}_G(1, q) = \kappa^{(q)}(\boldsymbol{u}). \tag{3.24}$$

### 3.5.2 Monomial positivity

Here, we prove the following theorem implying positivity of LLT cumulants for arbitrary $r$-colored LLT graphs in the quasi-symmetric monomial basis (this is a refinement of the main result from [14]):

**Theorem 3.7.** Let $(G, f)$ be an $r$-colored LLT graph and denote $E_d = E_d(G)$. Then:

$$\kappa_{\mathrm{LLT}}(G, f)(q) = \sum_{\substack{E \subseteq E_d \\ \hat{G}^E\, f\text{-connected}}} \mathcal{I}_{(\hat{G}^E)_f}(q)\, \mathrm{LLT}(\tilde{G}^E)(q), \tag{3.25}$$

where $\hat{G}^E$ is obtained from $\tilde{G}^E$ by removing all the edges of type II (i.e. $E(\hat{G}^E) = E(\tilde{G}^E) \setminus E_2(\tilde{G}^E)$).

*Proof.* We first realize that $G_\pi$ is obtained from $G$ by removing all the double edges connecting vertices with colors lying in different blocks of $\pi$. Then, we apply local transformations $\pi'_{\vec{e}}(G_\pi)$ for all $\vec{e} \in E_d(G_\pi)$ and $\pi'''_{\vec{e}}(G_\pi) := (G_\pi)_{\vec{e} \to \overrightarrow{e_1}} + (G_\pi)_{\vec{e} \to \overrightarrow{e_2}}$ for all $\vec{e} \in E_d(G) \setminus E_d(G_\pi)$. Notice also that, $\mathrm{LLT}(G) = \mathrm{LLT}(\pi'''_{\vec{e}}(G))$ for any orientation $\vec{e}$ of $e \notin E(G)$.

Let $\hat{G}^E$ be a graph obtained from $\tilde{G}^E$ by removing all the edges of type II and we recall that $(\hat{G}^E)_f$ is a graph obtained from $\hat{G}^E$ by identifying vertices of the same color, i.e. $v \sim w$ if $f(v) = f(w)$. Note that the vertex set of $(\hat{G}^E)_f$ is equal to $[r]$ and $|E_B|$ in the previous formula is equal to the number of edges of $(\hat{G}^E)_f$ with both endpoints belonging to $B$ (that we denote by $e_B$ to be consistent with the previous notation). Therefore, following (3.16), we end up with the formula

$$\begin{aligned}\kappa_{\mathrm{LLT}}(G, f) &= (q-1)^{1-r} \sum_{E \subseteq E_d(G)} \mathrm{LLT}(\tilde{G}^E) \Bigg( \sum_{\pi \in \mathcal{P}([r])} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{B \in \pi} q^{|E_B|} \Bigg) \\ &= \sum_{E \subseteq E_d(G)} \kappa^{(q)}_{[r]}(\boldsymbol{u})\, \mathrm{LLT}(\tilde{G}^E),\end{aligned}$$

where $u_B := q^{e_B}$. This can be rewritten as

$$\sum_{E \subseteq E_d(G)} \mathcal{I}_{(\hat{G}^E)_f}(q)\, \mathrm{LLT}(\tilde{G}^E)(q)$$

thanks to Lemma 3.6. Finally, $\mathcal{I}_{(\hat{G}^E)_f}(q) = 0$ whenever $(\hat{G}^E)_f$ is not connected (because disconnected graphs have no spanning trees), which is the very definition of being $f$-connected for $\hat{G}^E$. This finishes the proof. $\square$

### 3.5.3 Fundamental quasisymmetric functions and Conjecture 3 for hooks

For any non-negative integer $n$ and a subset $A \subseteq [n-1]$, we define the *fundamental quasisymmetric function* $F_{n,A}(x)$ to be the expression

$$F_{n,A}(x) := \sum_{\substack{i_1 \leq \cdots \leq i_n \\ j \in A \Longrightarrow i_j < i_{j+1}}} x_{i_1} \dots x_{i_n}.$$

For a standard tableau $T \in \mathrm{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ on a sequence $\boldsymbol{\lambda}/\boldsymbol{\mu}$, we define the set of descents $\mathrm{Des}(T)$ of $T$ (note that this is not the same as the set of descents of a tuple of skew shapes $\boldsymbol{\lambda}/\boldsymbol{\mu}$, which appeared in Eq. (2.5)) as the set of $i \in [n]$ such that $\tilde{c}(T^{-1}(i+1)) < \tilde{c}(T^{-1}(i))$.

In [27], Haglund, Haiman and Loehr implicitly[1] proved the following formula for the expansion of LLT polynomials in the fundamental quasisymmetric functions.

**Theorem 3.8** ([27])**.** For a sequence of skew shapes $\boldsymbol{\lambda}/\boldsymbol{\mu}$ with $|\boldsymbol{\lambda}/\boldsymbol{\mu}| = n$, we have

$$\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu}) = \sum_{T \in \mathrm{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu})} q^{\mathrm{inv}(T)} F_{n,\mathrm{Des}(T)}(x). \tag{3.26}$$

What is more, we can obtain a similar result in our language and notation.

**Corollary 2.** *For any $r$-colored tuple $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$ of size $n$ and for any set partition $\pi \in \mathcal{P}([r])$, we have*

$$\prod_{B \in \pi} \mathrm{LLT}((\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B) = \sum_{T \in \mathrm{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu})} q^{\mathrm{inv}_\pi(T)} F_{n,\mathrm{Des}(T)}(x), \tag{3.27}$$

*where $\mathrm{inv}_\pi(T)$ denotes the number of inversions in $T$ with both boxes in the same block of $\pi$.*

*Proof.* The result is a straightforward application of the arguments used in [27]. □

Applying the same proof as in Theorem 3.7 to (3.27), we obtain the following result (see also [14, Section 5] for an analogous argument applied to Macdonald cumulants):

**Theorem 3.9.** Let $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$ be an $r$-colored sequence of skew shapes of size $n$. Then:

$$\kappa_{\mathrm{LLT}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)(q) = \sum_{T \in \mathrm{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu})} \sum \mathcal{I}_{(\widehat{G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}}^{E^T})_f}(q) F_{n,\mathrm{Des}(T)}(x), \tag{3.28}$$

where the second sum runs over all subsets $E^T \subseteq E_d(G_{\boldsymbol{\lambda}/\boldsymbol{\mu}})$ for which $\widehat{G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}}^{E^T}$ is $f$-connected and $T(i) > T(j)$ whenever $(i,j) \in E(\widehat{G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}}^{E^T})$.

In [14], we were able to find an explicit formula for the coefficients of Schur symmetric functions indexed by *hooks*, i.e., partitions of the form $(k, 1^{n-k})$, in Macdonald cumulants, thanks to the arguments from [27]. Here, we will use a very nice theorem of Egge, Loehr and Warrington [16] which gives a combinatorial description of Schur coefficients of any symmetric function when given an expansion in fundamental quasisymmetric functions.

[1] instead of LLT polynomials they expanded Macdonald polynomials into fundamental quasisymmetric functions, but their arguments can be directly applied to LLT polynomials yielding (3.26)

**Theorem 3.10** ([16]). Suppose that

$$\sum_{\lambda \vdash n} c_\lambda s_\lambda = \sum_{\alpha \models n} d_\alpha F_{n,A(\alpha)},$$

where $A(\alpha) = (\alpha_1, \alpha_1+\alpha_2, \dots, \sum_{i=1}^{\ell(\alpha)-1} \alpha_i)$. Then we have $c_{(k,1^{n-k})} = d_{(k,1^{n-k})}$ for all $1 \leq k \leq n$.

The original result gives a description of the coefficients $c_\lambda$ for a general $\lambda \vdash n$. However, since we only need the case in the statement (i.e., when $\lambda$ is a *hook*), we refer interested readers to [16] for the general version and the details on the combinatorial gadgets used in the proof.

The following theorem is an immediate corollary from Theorem 3.9 and Theorem 3.10:

**Theorem 3.11.** Let $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$ be an $r$-colored sequence of skew shapes of size $n$. Then for any $1 \leq k \leq n$

$$[s_{(k,1^{n-k})}]\kappa_{\mathrm{LLT}}(\boldsymbol{\lambda}/\boldsymbol{\mu}) = \sum_{\substack{T \in \mathrm{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu}) \\ \mathrm{Des}(T)=\{k,k+1,\dots,n-1\}}} \sum \mathcal{I}_{(\widehat{G_{\boldsymbol{\lambda}}}^{E^T})_f}(q),$$

where the second sum runs over all subsets $E^T \subseteq E_d(G_{\boldsymbol{\lambda}/\boldsymbol{\mu}})$ for which $\widehat{G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}}^{E^T}$ is $f$-connected and $T(i) > T(j)$ whenever $(i,j) \in E(\widehat{G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}}^{E^T})$.

### 3.5.4 e-positivity

Recall that $(e_\lambda)_\lambda$ denotes the basis of elementary symmetric functions. $e$-positivity of a given symmetric function $f$ is a stronger property than Schur-positivity and it suggests a specific interpretation of the function $f$ in terms of the representation theory of the symmetric group, and in algebro-geometric context.

This observation recently generated a lot of research in studying $e$-positive symmetric functions, and after a series of conjectures [6, 4, 19], it was clear that $e$-positivity of a big class of symmetric functions would be a consequence of $e$-positivity for *vertical-strip LLT polynomials* after the shift $q \to q+1$, i.e. for $\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu})(q+1)$ where $\lambda_i/\mu_i = (1^{n_i+k_i})/(1^{k_i})$ for each $1 \leq i \leq \ell(\boldsymbol{\lambda}/\boldsymbol{\mu})$ and some nonnegative integers $n_i, k_i$. An explicit combinatorial formula for the coefficients of vertical-strip LLT polynomials in the basis of elementary functions was independently conjectured in [19, 3][2] and shortly afterwards the positivity (without proving the combinatorial interpretation) was proved in [11] and subsequently [5] finalized the picture by proving the combinatorial interpretation. In the following, we reformulate this combinatorial interpretation in our current framework.

Let $\boldsymbol{\lambda}/\boldsymbol{\mu}$ be a tuple of vertical-strips and let $G = G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ be the associated LLT-graph. We recall that a vertex $v \in V(G)$ is associated with a box $\square(v) \in \boldsymbol{\lambda}/\boldsymbol{\mu}$ and the vertices are naturally labeled by the shifted contents of the corresponding boxes $\tilde{c}(v) := \tilde{c}(\square(v))$. Fix $E \subseteq E_d(G)$ and define $G' = G^E$. Since $G'$ is a directed graph (note that the condition that $\boldsymbol{\lambda}/\boldsymbol{\mu}$ is a tuple of vertical strips

[2]in fact, these interpretations are not identical, since the authors use slightly different framework in their works, but it is possible to show that they are equivalent

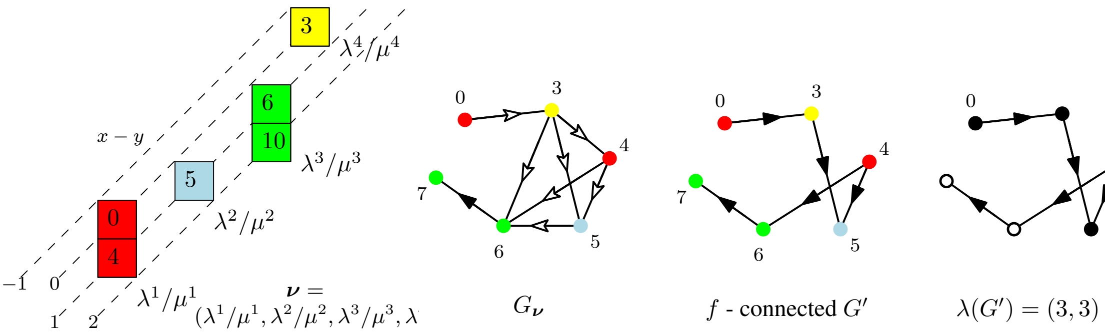



Figure 3.3: A tuple $\boldsymbol{\lambda}$ of vertical strips and the associated LLT graph $G_{\boldsymbol{\lambda}}$. The labels of vertices are the shifted contents of the corresponding boxes. Graph $G' = G^E$ for $E = \{(0,3),(3,5),(4,5),(4,6)\}$ is $f$-connected, where $f(i) = i$. In the last picture, the displayed labels are the labels of the sources of $G'$ and the corresponding two equivalence classes $\{0,3,5\}$ and $\{4,6,7\}$ are depicted by the whole and the empty vertices, respectively.

implies that $G'$ has only edges of type I), some of the vertices of $G'$ have only outgoing edges – such vertices are called *sources*. We define the following equivalence relation on the set of vertices $V(G')$: the vertices $v \sim w$ are in the same equivalence class if the source $\theta(v)$ with the smallest label from which there exists a directed path to $v$ is the same as the source $\theta(w)$ with the smallest label from which there exists a directed path to $w$. The partition $\lambda(G')$ is defined as the partition whose parts are sizes of the equivalence classes in this relation. See Section 3.5.4 for an example.

**Theorem 3.12.** [5] Let $\boldsymbol{\lambda}/\boldsymbol{\mu}$ be a tuple of vertical-strips and let $G = G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ be the associated LLT-graph. Then

$$\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu})(q+1) = \sum_{E \subseteq E_d(G)} q^{|E|} e_{\lambda(G^E)}. \tag{3.29}$$

In the following, we show that the vertical-strip LLT cumulants preserve $e$-positivity, which refines Theorem 3.12, but most importantly shows that $e$-positivity of vertical-strips LLT polynomials naturally decomposes into $f$-connected components, each corresponding to the vertical-strip LLT cumulant. In other terms, heuristically, the e-positivity of vertical-strip LLT polynomials is “built” from $e$-positivity of LLT cumulants, which naturally decompose LLT polynomials from the graph-coloring point of view.

**Theorem 3.13.** Let $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$ be an $r$-colored tuple of vertical-strips and let $G = G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ be the associated LLT-graph. Then

$$\kappa_{\mathrm{LLT}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)(q+1) = \sum_{\substack{E \subseteq E_d \\ G^E\ f\text{-connected}}} q^{|E|+1-r} e_{\lambda(G^E)}. \tag{3.30}$$

*Proof.* We recall that the $q$-partial cumulant of the family $(\boldsymbol{u})$ is defined by the formula (3.1). One can invert this formula in order to express $u_I$ in terms of the $q$-partial cumulants:

$$u_I = \sum_{\pi \in \mathcal{P}(I)} (q-1)^{|I|-|\pi|} \prod_{B \in \pi} \kappa_B^{(q)}(\boldsymbol{u}).$$

Applying this to our setting, we obtain that for any $r \geq 1$ and for any $r$-colored tuple $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$, one has

$$LLT(\boldsymbol{\lambda}/\boldsymbol{\mu})(q+1) = \sum_{\pi \in \mathcal{P}([r])} q^{r-|\pi|} \prod_{B \in \pi} \kappa_{\mathrm{LLT}}((\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B, f|_B)(q+1),$$

where $f|_B$ is the $|B|$-coloring of $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B$ obtained from $f$ by restricting it to the preimage of $B$, i.e., $f|_B \colon f^{-1}(B) \to B$.

We will prove (3.30) by induction on $r$. For $r = 1$, the LHS of (3.30) is equal to $\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu})(q+1)$, while the RHS of (3.30) coincides with the RHS of (3.29), because every 1-colored graph is trivially $f$-connected.

Let $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$ be an $r$-colored tuple of vertical-strips with $r > 1$. Let $G' = G^E$ for some $E \subseteq E_d(G)$. Note that decomposing $G'$ into $f$-connected components, we find a set-partition $\pi \in \mathcal{P}([r])$ such that each $f$-connected component has a vertex set $V_B \coloneqq \{v \in V(G') \text{ colored by } b \in B\}$ for some $B \in \pi$. Therefore, we can rewrite (3.29) as follows:

$$\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu})(q+1) = \sum_{\pi \in \mathcal{P}([r])} \prod_{B \in \pi} \Bigg( \sum_{\substack{E_B \subseteq E_d(G_B) \\ G_B^{E_B}\ f|_B\text{-connected}}} q^{|E_B|} \Bigg) e_{\lambda(\bigoplus_B G_B^{E_B})}.$$

Notice also that $e_{\lambda(\bigoplus_B G|_{V_B}^{E_B})} = \prod_{B \in \pi} e_{\lambda(G|_{V_B}^{E_B})}$, which is immediate from the definition of $\lambda(G')$. Indeed, the whole equivalence class has to be contained in the connected component of $G$, which is further contained in the $f$-connected component. Using the obvious identity

$$q^{r-|\pi|} = \prod_{B \in \pi} q^{|B|-1},$$

we obtain

$$\kappa_{\mathrm{LLT}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)(q+1) = \sum_{\pi \in \mathcal{P}([r])} q^{1-|\pi|} \prod_{B \in \pi} \Bigg( \sum_{\substack{E_B \subseteq E_d(G_B)\colon \\ G_B^{E_B} \text{ is } f|_B\text{-connected}}} q^{|E_B|} \Bigg) e_{\lambda(\bigoplus_B G_B^{E_B})}$$
$$- \sum_{\substack{\pi \in \mathcal{P}([r]) \\ \pi \neq \{[r]\}}} q^{r-|\pi|} \prod_{B \in \pi} \kappa_{\mathrm{LLT}}((\boldsymbol{\lambda}/\boldsymbol{\mu}, f)^B, f|_B)(q+1) = \sum_{\substack{E \subseteq E_d \\ G^E\ f\text{-connected}}} q^{|E|+1-r} e_{\lambda(G^E)},$$

where the last equality follows by the inductive hypothesis, and the proof is finished. $\square$

## 3.6 Concluding remarks and questions

We conclude the chapter by proving Conjecture 3 for some special cases and stating some more general open questions.

We start by showing that Conjecture 3 holds true when $\ell(\boldsymbol{\lambda}/\boldsymbol{\mu}) = 2$.

**Proposition 3.14.** Let $\boldsymbol{\lambda}/\boldsymbol{\mu} = ((\lambda^1/\mu^1, \lambda^2/\mu^2), f)$ be an $r$-colored pair of skew Young diagrams. Then, for every partition $\nu$ the coefficient

$$[s_\nu]\kappa_{\mathrm{LLT}^{\mathrm{cospin}}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f) \in \mathbb{Z}_{\geq 0}[q]$$

is a polynomial in $q$ with nonnegative integer coefficients.

*Proof.* We know that LLT polynomials are Schur positive, i.e

$$\mathrm{LLT}^{\mathrm{cospin}}(\boldsymbol{\lambda}/\boldsymbol{\mu})(q) = \sum_\nu c^\nu_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q) s_\nu,$$

where $c^\nu_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q) = \sum_{i=0}^{d^\nu_{\boldsymbol{\lambda}/\boldsymbol{\mu}}} c^{\nu;i}_{\boldsymbol{\lambda}/\boldsymbol{\mu}} q^i \in \mathbb{Z}_{\geq 0}[q]$ and we know that

$$\mathrm{LLT}^{\mathrm{cospin}}(\lambda^1/\mu^1)(q)\,\mathrm{LLT}^{\mathrm{cospin}}(\lambda^2/\mu^2)(q) = \mathrm{LLT}^{\mathrm{cospin}}(\boldsymbol{\lambda}/\boldsymbol{\mu})(1).$$

Therefore, the case of 2-coloring gives us

$$\kappa_{\mathrm{LLT}^{\mathrm{cospin}}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f) = \sum_\nu \sum_{i=1}^{d^\nu_{\boldsymbol{\lambda}/\boldsymbol{\mu}}} c^{\nu;i}_{\boldsymbol{\lambda}/\boldsymbol{\mu}} [i]_q s_\nu.$$

Since the LLT cumulant of 1-colored tuple is simply an LLT polynomial (which is Schur positive by the result of Grojnowski and Haiman [25]) and there are no other $r$-colorings of a pair of skew partitions, the proof is finished. □

**Remark 5.** *Note that in this case, the coefficient $[s_\nu]\kappa_{\mathrm{LLT}^{\mathrm{cospin}}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$ is explicit assuming that the coefficient $[s_\nu]\,\mathrm{LLT}^{\mathrm{cospin}}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ is known. In our setting, this coefficient was described combinatorially in terms of inversions of Yamanouchi tableaux by Roberts [52], which, in effect, provides also the combinatorial interpretation of the coefficient $[s_\nu]\kappa_{\mathrm{LLT}^{\mathrm{cospin}}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$.*

An explicit expression for $\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ in the Schur basis exists also for $\ell(\boldsymbol{\lambda}/\boldsymbol{\mu}) = 3$ due to Blasiak [7] but it is much more complicated and, as noticed by Blasiak, there are serious difficulties in going beyond the case $\ell(\boldsymbol{\nu}) = 3$. Let us recover Blasiak's result here [7, Corollary 4.3], so that we can state our conjecture connected to its cumulant counterpart.

Let $\boldsymbol{\lambda}/\boldsymbol{\mu} = (\lambda^1/\mu^1, \lambda^2/\mu^2, \lambda^3/\mu^3)$. Blasiak proved that

$$\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu})(q) = \sum_\nu c^\nu_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q) s_\nu, \qquad \text{where} \qquad c^\nu_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q) = \sum_{\substack{T \in \mathrm{RSST}(\nu) \\ \mathrm{Des}'_3(T) = D'(\boldsymbol{\lambda}/\boldsymbol{\mu}) \\ \tilde{c}(\boldsymbol{\lambda}/\boldsymbol{\mu}) \text{ - entries of } T}} q^{\mathrm{inv}'_3(T)}, \tag{3.31}$$

and

- $\mathrm{RSST}(\nu)$ is the set of *restricted square strict tableaux* of shape $\nu$, i.e., fillings of $\nu$ whose columns strictly increase upwards, rows strictly increase rightwards, and the filling of the cell $(x, y)$ is smaller by at least 3 than that of $(x', y')$ with $x' > x$ and $y' > y$;

- $\mathrm{Des}'_3(T)$ is the multiset of pairs $(T(x,y), T(x',y'))$ with $(x,y),(x',y') \in \mathrm{sh}(T) = \nu$, such that $T(x,y) - T(x',y') = 3$, and either $y > y'$ and $x \leq x'$, or $x = x'+1$, $y = y'+1$, and $T(x',y) = T(x,y) - 1$;
- $D'(\boldsymbol{\lambda}/\boldsymbol{\mu})$ is the multiset of pairs $(\tilde{c}(x,y), \tilde{c}(x',y'))$ with $(x,y),(x',y') \in \boldsymbol{\lambda}/\boldsymbol{\mu}$, such that $\tilde{c}(x,y) = \tilde{c}(x',y') + 3$ and $y < y'$ and $x \leq x'$;
- $\tilde{c}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ is the sequence of shifted contents of $\boldsymbol{\lambda}/\boldsymbol{\mu}$; and
- $\mathrm{inv}'_3(T)$ is the number of pairs $((x,y),(x',y'))$ with $(x,y),(x',y') \in \mathrm{sh}(T)$ with $0 < T(x,y) - T(x',y') < 3$, such that $y > y'$ and $x \leq x'$.

Note that the sets $\mathrm{Des}'_3(T)$ and $D'(\boldsymbol{\lambda}/\boldsymbol{\mu})$ are indeed multisets. For instance, for $\boldsymbol{\lambda}/\boldsymbol{\mu} = ((3,3,3),(1),(1))$, we have

$$D'(\boldsymbol{\lambda}/\boldsymbol{\mu}) = \{(6,3),(3,0),(3,0),(3,0),(0,-3),(0,-3),(0,-3),(-3,-6)\}.$$

**Example.** *Let $\lambda^1/\mu^1 = \lambda^3/\mu^3 = (1,1)$ and $\lambda^2/\mu^2 = (2,2)/(2)$ and consider $[s_{(3,2,1)}]\,\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ for $\boldsymbol{\lambda}/\boldsymbol{\mu} = (\lambda^1/\mu^1, \lambda^2/\mu^2, \lambda^3/\mu^3)$. According to (3.31), it is counted by restricted square strict tableaux of shape $(3,2,1)$ with some additional constraints. On the left hand side of Fig. 3.4, we show $\boldsymbol{\lambda}/\boldsymbol{\mu}$ with its shifted contents and we give an example of a restricted square strict tableau $T$ of shape $(3,2,1)$, which satisfies the constraint $\mathrm{Des}'_3(T) = D'(\boldsymbol{\lambda}/\boldsymbol{\mu})$. We colored the boxes of $\lambda^i/\mu^i$, therefore the pairs counting $\mathrm{inv}'_3(T)$ can be represented as the edges of a graph on three vertices (which is shown on two drawings on the right hand side).*

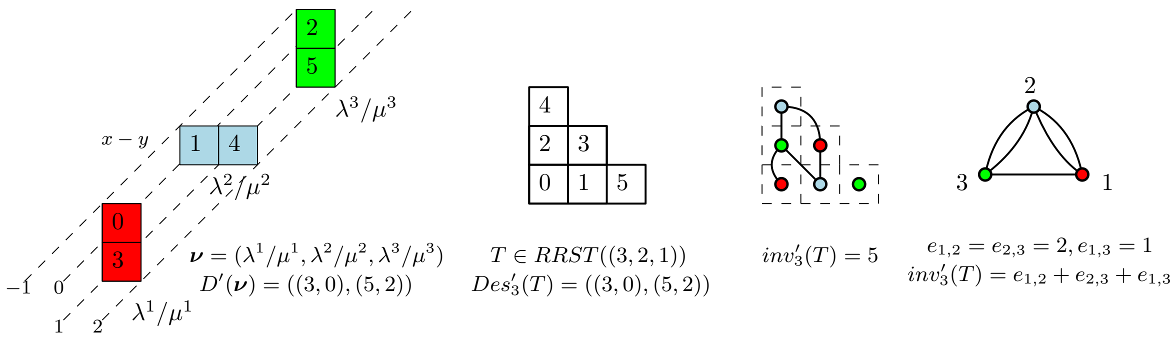



Figure 3.4: Restricted square strict tableau corresponding to Schur-expansion of LLT polynomials of three skew shapes.

Using the notation from Fig. 3.4, let $e_{i,j}(T)$ denote the number of pairs $(\square, \square')$ contributing to $\mathrm{inv}'_3(T)$ with $T(\square) \equiv i$ and $T(\square') \equiv j$ modulo 3, so that

$$\mathrm{inv}'_3(T) = e_{1,2}(T) + e_{1,3}(T) + e_{2,3}(T).$$

We believe that the following is true

**Conjecture 4.** *For any triple of skew diagrams $\boldsymbol{\lambda}/\boldsymbol{\mu} = (\lambda^1/\mu^1, \lambda^2/\mu^2, \lambda^3/\mu^3)$ and every triple $\{i, j, k\} = \{1, 2, 3\}$ with $i < j$, we have*

$$\mathrm{LLT}(\lambda^i/\mu^i, \lambda^j/\mu^j)(q) \cdot \mathrm{LLT}(\lambda^k/\mu^k)(q) = \sum_{\nu} \left( \sum_{\substack{T \in \mathrm{RSST}(\nu) \\ \mathrm{Des}'_3(T) = D'(\boldsymbol{\lambda}/\boldsymbol{\mu}) \\ c(\boldsymbol{\lambda}/\boldsymbol{\mu}) \text{ - entries of } T}} q^{e_{i,j}(T)} \right) s_\nu. \tag{3.32}$$

**Corollary 3.** *Assuming Conjecture 4, Conjecture 3 holds true for $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$ a triple of skew shapes.*

*Proof.* The proof follows the same argument as the one used in Theorem 3.7 to show that

$$[s_\nu]\kappa_{\mathrm{LLT}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f) = \sum_{\substack{T \in \mathrm{RSST}(\nu) \\ \mathrm{Des}'_3(T) = D'(\boldsymbol{\lambda}/\boldsymbol{\mu}) \\ c(\boldsymbol{\lambda}/\boldsymbol{\mu}) \text{ - entries of } T}} \mathcal{I}_{(G^T)_f}(q),$$

where $G^T$ is an $f$-colored graph whose vertices are entries of $T$ and we connect pairs contributing to $\mathrm{inv}'_3(T)$. □

Note that the above argument works for any $r$-colored tuple of shapes $(\boldsymbol{\lambda}/\boldsymbol{\mu}, f)$ and thus, Conjecture 3 suggests the following interesting structure of the coefficients of LLT-polynomials in the Schur expansion.

**Problem 1.** *Let $\boldsymbol{\lambda}/\boldsymbol{\mu} = (\lambda^1/\mu^1, \ldots, \lambda^r/\mu^r)$ be an $r$-tuple of skew Young diagrams. Is it true that for any partition $\nu$ there exists a class of graphs $\mathcal{G}_\nu^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ with the set of vertices $[r]$ such that for any set-partition $\pi \in \mathcal{P}([r])$ one has*

$$[s_\nu] \prod_{B \in \pi} \mathrm{LLT}^{\mathrm{cospin}}((\boldsymbol{\lambda}/\boldsymbol{\mu})^B) = \sum_{G \in \mathcal{G}_\nu^{\boldsymbol{\lambda}/\boldsymbol{\mu}}} q^{\sum_{B \in \pi} e_B},$$

*where $(\boldsymbol{\lambda}/\boldsymbol{\mu})^B := (\boldsymbol{\lambda}/\boldsymbol{\mu}, \mathrm{id}_{[r]})^B$ and $\mathrm{id}_{[r]} \colon [r] \to [r]$ is the identity function?*

Note that the affirmative answer for this problem implies Conjecture 3 providing its combinatorial interpretation:

$$[s_\nu]\kappa_{\mathrm{LLT}^{\mathrm{cospin}}}(\boldsymbol{\lambda}/\boldsymbol{\mu}, f) = \sum_{G \in \mathcal{G}_\nu^{\boldsymbol{\lambda}/\boldsymbol{\mu}}} \mathcal{I}_{(G)_f}(q).$$

In the next section, we show that Problem 1 has an affirmative answer in some special cases and thus, Conjecture 3 holds true for them.

### 3.6.1 Unicellular LLT and melting lollipops

A *Schröder path* of length $n$ is a path from $(0, 0)$ to $(n, n)$ composed by steps $\uparrow = (0, 1)$, $\rightarrow = (1, 0)$, and $\nearrow = (1, 1)$ (referred to as *north*, *east*, and *diagonal steps*, respectively), which stays above the main diagonal (i.e., it can touch it, but the diagonal steps lie strictly above it). Denote by $(i, j)$ the

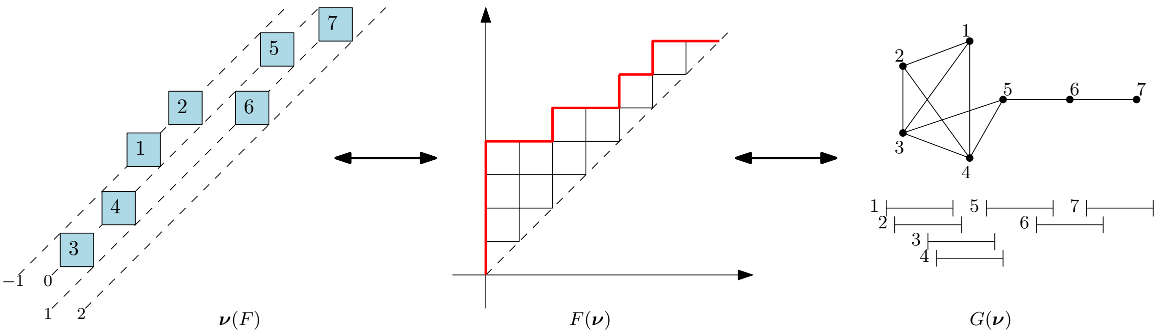



Figure 3.5: The correspondence between unicellular LLT polynomials, Dyck paths and unit interval graphs. The graph $G(\boldsymbol{\lambda}/\boldsymbol{\mu})$ on the right is the melting lollipop graph $L^{(2)}_{(5,2)}$ and we display the arrangment of unit intervals which realizes it as the unit interval graph.

coordinates of the $1 \times 1$ box with upper right vertex in $(i, j)$. It is well known [26] that the vertical-shape LLT polynomials of homogenous degree $n$ are in bijection with Schröder paths of length $n$: start from an $r$-tuple of vertical shapes $\boldsymbol{\lambda}/\boldsymbol{\mu}$ of size $n$, label its boxes by their shifted contents and standardize them, i.e. replace them (in the unique way) by labels in $[n]$ such that the order of new labels is the same as the order of shifted contents. Now construct a Schröder path $F(\boldsymbol{\lambda}/\boldsymbol{\mu})$ such that the box $(i, j)$ lies below the path if and only if the entry $i$ attacks the entry $j$ in $\boldsymbol{\lambda}/\boldsymbol{\mu}$ and the box $(i, j)$ lies on the diagonal step if the entry $j$ lies directly below the entry $i$. This procedure is clearly invertible and we denote by $\boldsymbol{\lambda}/\boldsymbol{\mu}(F)$ the tuple of vertical strips associated with the Schröder path $F$ (see the left side of Fig. 3.5 and consult [26] for more details).

A special case of a Schröder path is a *Dyck path*, that is a path with no diagonal steps. The corresponding $r$-tuple of vertical shapes $\boldsymbol{\lambda}/\boldsymbol{\mu}$ of size $n$ is a sequence of $r$ single boxes (i.e. $r = n$) and its LLT polynomial is called *unicellular*. It is remarkable that the LLT graphs associated with unicellular LLT polynomials are precisely *unit interval graphs*, i.e. they can be realized as the intersection graphs of $n$ unit intervals on the line (see the right side of Fig. 3.5).

Note that for every unit interval graph $G$ on $n$ vertices one has

$$\mathrm{LLT}(G)(1) = e_1^n = \sum_{\lambda \vdash n} \sum_{T \in \mathrm{SYT}(\lambda)} s_\lambda.$$

Therefore, it is natural to look for a statistic $a_G : \mathrm{SYT} \to \mathbb{Z}_{\geq 0}$ such that

$$[s_\lambda]\,\mathrm{LLT}(G)(q) = \sum_{T \in \mathrm{SYT}(\lambda)} q^{a_G(T)}.$$

Recall that the descent set $\mathrm{Des}(T)$ of a standard Young tableau $T \in \mathrm{SYT}(\lambda)$ is given by the values $i \in [n-1]$ for which the entry $i+1$ lies in $T$ in a row above the entry $i$[3] and define

$$\overleftarrow{\mathrm{Des}(T)} := \{n+1-i \colon i \in \mathrm{Des}(T)\}.$$

[3] it is easy to check that this definition coincides with the previous definition of $\mathrm{Des}(T)$ in the special case of $\boldsymbol{\lambda}/\boldsymbol{\mu} = (\lambda)$ and $T \in \mathrm{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu})$

**Definition 3.6.** Let $m \geq 1, n$ be nonnegative integers and $0 \leq k \leq m-1$. A *melting lollipop* $L^{(k)}_{(m,n)}$ is a graph with the vertex set $[m+n]$, built by joining the complete graph on vertices $[m]$ with the path on vertices $\{m, \dots, m+n\}$ (with edges of the form $(i, i+1)$) and erasing edges $(1,m), (2,m), \dots, (k,m)$. The unit interval graph depicted on Fig. 3.5 is the melting lollipop $L^{(2)}_{(5,2)}$.

Recently Huh, Nam and Yoo proved the following theorem [32]:

**Theorem 3.15.** [32] Let $\mathcal{F}_n$ be a family of unit interval graphs with $n$ vertices such that

$$\mathrm{LLT}(G)(q) = \sum_{\lambda \vdash n} \sum_{T \in \mathrm{SYT}(\lambda)} q^{\sum_{i \in \overleftarrow{\mathrm{Des}(T)}} \deg^G_{\mathrm{in}}(i)} s_\lambda$$

for each $G \in \mathcal{F}_n$ (here $\deg^G_{\mathrm{in}}(i)$ denotes the number of edges in $G$ incoming to the vertex $i$). Then $\mathcal{F}_n$ contains melting lollipops and their disjoint unions.

Melting lollipops contain two extremal cases for which Theorem 3.15 is a classical result: the complete graph $K_n = L^{(0)}_{(n,0)}$ and the path graph $P_n = L^{(0)}_{(1,n-1)} = L^{(0)}_{(2,n-2)}$.

**Theorem 3.16.** Let $G$ be a melting lollipop graph with $r$ vertices. Then for every set-partition $\pi \in \mathcal{P}([r])$, one has

$$[s_\nu] \prod_{B \in \pi} \mathrm{LLT}^{\mathrm{cospin}}(G|_B) = \sum_{T \in \mathrm{SYT}(\nu)} q^{\sum_{B \in \pi} e_B\left(G_{\mathrm{in}}^{\overleftarrow{\mathrm{Des}(T)}}\right)},$$

where $G^A_{\mathrm{in}}$ is a graph obtained from $G$ by removing all the edges which are not incoming to vertices in $A \subset V$. In particular, Problem 1 and Conjecture 3 have an affirmative answer in this case and

$$[s_\nu]\kappa_{\mathrm{LLT}^{\mathrm{cospin}}}(G, f) = \sum_{T \in \mathrm{SYT}(\nu)} \mathcal{I}_{\left(G_{\mathrm{in}}^{\overleftarrow{\mathrm{Des}(T)}}\right)_f}(q).$$

*Proof.* It is enough to notice that

- for every set-partition $\pi \in \mathcal{P}([r])$ the graph $G_\pi := \bigoplus_{B \in \pi} G|_{V_B}$ is a disjoint union of melting lollipops so that

$$\prod_{B \in \pi} \mathrm{LLT}(G|_{V_B})(q) = \mathrm{LLT}(G_\pi)(q) = \sum_{\lambda \vdash n} \sum_{T \in \mathrm{SYT}(\lambda)} q^{\sum_{i \in \overleftarrow{\mathrm{Des}(T)}} \deg^{G_\pi}_{\mathrm{in}}(i)} s_\lambda;$$

- the identity

$$\sum_{i \in A} \deg^G_{\mathrm{in}}(i) = |E(G^A_{\mathrm{in}})|$$

follows directly from the construction of $G^A_{\mathrm{in}}$.

□

**Remark 6.** *Note that the class $\mathcal{F}_n$ is strictly smaller than the class of unit interval graphs on $n$ vertices which can be seen already for $n = 4$: the unit interval graph $G = (V = [4], E)$ with $E = \{(1,2), (2,3), (2,4), (3,4)\}$ does not belong to $\mathcal{F}_n$. On the other hand, we were not able to find any graph which belongs to $\mathcal{F}_n$ and is not a disjoint union of melting lollipops, and it is tempting to conjecture that these two classes of graphs coincide.*

# Chapter 4

# A combinatorial formula for LLT cumulants of melting lollipops

This chapter is a modified version of [42]: an unpublished paper which is available in the public repository at arXiv.org.

**Abstract:** We prove a combinatorial formula for LLT cumulants of melting lollipops as a positive combination of LLT polynomials indexed by spanning trees. The result gives an affirmative answer to a question from [12] for this class of unicellular LLT cumulants, and gives an independent proof of their Schur-positivity. In the special case of the complete graph, we also express the formula in terms of parking functions.

## 4.1 Introduction

Arguably, Conjecture 3 is difficult to prove in its general statement. Therefore, it is natural to first approach the problem in a special case and for us, that case is unicellular diagrams, i.e., when the (skew) shapes consist of a single cell. It turns out that such sequences correspond to interesting interpretations of the LLT cumulant and are connected with classical combinatorial objects. In fact, based on extensive computer simulations, we have noticed that it is always possible to find an expansion of the following special form:

$$\kappa(\boldsymbol{\lambda}/\boldsymbol{\mu}) = \sum_{T} \mathrm{LLT}(\nu(T)), \tag{4.1}$$

where $\boldsymbol{\lambda}/\boldsymbol{\mu}$ is a sequence of unicellular shapes and the sum runs over some spanning trees of $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ and $\nu(T)$ denotes a sequence of vertical-strip shapes corresponding to $T$ (see Section 4.2 for details).

The main result of this chapter is Theorem 4.1, which gives the proof of (4.1) in the special case of $\boldsymbol{\lambda}/\boldsymbol{\mu}$ corresponding to a melting lollipop (see Definition 3.6). In our setting, melting lollipops appear as LLT graphs in the sense of Definition 3.3. In this case, we have the following formula.

**Theorem 4.1.** Let $\boldsymbol{\lambda}/\boldsymbol{\mu}$ be a sequence of $l+m$ unicellular diagrams corresponding to a melting lollipop $L^{(k)}_{(l,m)}$. Then

$$\kappa(\boldsymbol{\lambda}/\boldsymbol{\mu}) = \sum_{T \subseteq L^{(k)}_{(l,m)}} \mathrm{LLT}(\nu(T)), \tag{4.2}$$

where the sum runs over all spanning trees of $L^{(k)}_{(l,m)}$.

Note that thanks to the result of Grojnowski and Haiman [25], as an immediate consequence of the above theorem, we get Schur-positivity of LLT cumulants in the special case of $\boldsymbol{\lambda}/\boldsymbol{\mu}$ corresponding to a melting lollipop. It is an independent proof to the one presented in [12].

Moreover, if we apply the result to the case $k = l = 0$, i.e., when the melting lollipop is a complete graph, we can further express the formula in terms of parking functions.

**Corollary 4.** *Let $\boldsymbol{\lambda}/\boldsymbol{\mu} = ((1), \dots, (1))$ be a sequence of $m$ unicellular non-skew shapes. Then,*

$$\kappa(\boldsymbol{\lambda}/\boldsymbol{\mu}) = \sum_{T \subseteq K_m} \mathrm{LLT}(\nu(T)) = \sum_{f \in PF_{m-1}} \mathrm{LLT}(\mu(f)), \tag{4.3}$$

*where the first sum runs over all Cayley trees with $m$ vertices, the second over all parking functions on $m-1$ cars, and $\nu(T)$ and $\mu(f)$ are sequences of vertical-strip shapes associated to such objects (see Section 4.2 and Section 4.6).*

We would like to mention the similarity between (4.3) and the celebrated formula for the character of the diagonal coinvariant algebra (conjectured in [28] and proved in [8]). Both formulas are given as linear combinations of LLT polynomials associated to parking functions, but the LLT polynomials corresponding to a fixed parking function differ in the two. It would be interesting to find a representation-theoretical interpretation of LLT cumulants, which encourages further investigation.

## 4.2 Diagrams corresponding to spanning trees

Let $T$ be a plane rooted tree. Heuristically, we use an algorithm similar to depth-first search to construct consecutive shapes of $\nu(T)$ by deleting the maximal left-most paths from the root. To be precise, we decompose $V(T) = W_1 \cup \cdots \cup W_k$, where $W_i$-s are paths chosen according to the following rules.

Let $W_1$ be the geodesic path from the root to the left-most leaf of $T$. Next, suppose that we have already defined $W_1, \dots, W_l$. To define $W_{l+1}$, choose a vertex $v \in V(T) \setminus (W_1 \cup \cdots \cup W_l)$ following the depth-first search algorithm with respect to $W_1, \dots, W_l$. To be precise, $v$ is the vertex which has a neighbour in some $W_j$ with $1 \leq j \leq l$ maximal and $v$ is the left-most vertex with the maximal distance from the root for the $j$ determined above. We let $W_{l+1}$ be the left-most geodesic path from $v$ downwards to a leaf of $T$.

We will now show that the decomposition is well defined. Firstly, to see that $V(T) = W_1 \cup \cdots \cup W_k$, suppose for contradiction that there exists $v \in V(T) \setminus (W_1 \cup \cdots \cup W_k)$. Without loss of generality, assume that $v$ is the left-most vertex with that property. Then, $v$ cannot be the root (since the root is in $W_1$), so there exist vertices $v_1, \dots, v_l$ with $v_l = v$, $v_i$ the parent of $v_{i+1}$, $i = 1, \dots, l-1$, and $v_i$-s are not in $W_1 \cup \dots \cup W_k$. Next, if all vertices to the left of $v_1$ are contained in $W_1 \cup \dots \cup W_r$ for some $1 \leq r \leq k$, then, by the construction above, $v_1$ must be the first vertex of $W_{r+1}$.

Lastly, to see that the decomposition is unique, suppose that we have two decompositions $V(T) = W_1 \cup \dots \cup W_k = W_1' \cup \dots \cup W_r'$ and that $W_i = W_i'$ for $i = 1, \dots, l$. Then, by construction, the first vertex $v$ of $W_{l+1}$ is a neighbour of some $W_j$ with $1 \leq j \leq l$ maximal and similarly for the first vertex $v'$ of $W_{l+1}'$. What is more, both $v$ and $v'$ are the left-most vertices with the maximal distance from the root with those properties, which means that $v = v'$. Since there exists a unique left-most geodesic path from $v$ to a leaf of $T$, we must have $W_{l+1} = W_{l+1}'$.

**Definition 4.1.** With notation as above, denote by $\nu(T) = (\nu^1, \dots, \nu^k)$ the *LLT polynomial associated with the tree* $T$, where $\nu^i = (1^{s_i})/(1^{t_i})$, $i = 1, \dots, k$ with

1. $s_i = h(T) - d(W_i)$, where $h(T) = \max\{d(r, v) \mid v \text{ is a leaf in } T\}$ and $d(W) = \min\{d(r, v) \mid v \in W\}$ with $r$ the root of $T$ and $d(r, v)$ the vertex length of the path from $r$ to $v$, and

2. $t_i = s_i - |W_i|$.

**Remark 7.** *Note that the sequence $\nu(T)$ is indeed well defined. Firstly, we always have $s_i > 0$ simply by the definition of $h(T)$. Secondly, $t_i \geq 0$ since the path from the root to the first vertex of $W_i$ followed by $W_i$ is shorter or equal to the maximal path in $T$, and thus $h(T) \geq |W_i| + d(W_i)$, $i = 1, \dots, k$.*

### 4.2.1 Correspondence between spanning trees and plane rooted trees

Recall that the statements of Theorem 4.1 and Corollary 4 mention spanning trees of certain graphs with a set of integers as the vertex set rather than plane rooted trees. Fortunately, there is an obvious way to associate with such a graph $T$ a planar drawing in the feel of Definition 4.1. To be precise, we root $T$ in the vertex labeled by the smallest integer and draw the children of each vertex from left to right in an increasing order (see Fig. 4.1).

For simplicity of notation, let us write $\nu(T)$ for the sequence of vertical strips associated to the planar drawing of a tree $T$ with vertex set $[m]$ in the sense of Definition 4.1 (see Fig. 4.1). Clearly, such a map $\nu$ is not injective since there exist multiple labelings of a given plane rooted tree which satisfy the rules above. What is more, $\nu$ does not map onto the set of all sequences of vertical-strip shapes since, with notation as in Definition 4.1, $s_1 > s_i$ for $i > 1$. In fact, the image of $\nu$ has a particularly nice description in terms of Schröder paths and parking functions, which we now introduce.

## 4.3 Schröder paths

Recall that a *Schröder path* of length $n$ is a lattice path from $(0, 0)$ to $(m, m)$ with steps $n = (0, 1)$, $e = (1, 0)$, and $d = (1, 1)$ (referred to as *north*, *east*, and *diagonal steps*, respectively), which never

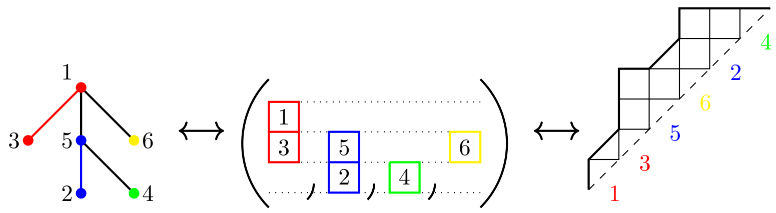


Figure 4.1: A spanning tree with its corresponding sequence of shapes and Schröder path.

falls below the main diagonal that joins the ends and has no $d$ steps on that diagonal. We denote by $(i, j)$ the coordinates of the $1 \times 1$ box with top right vertex in $(i, j)$.

For such a path $P$, we say that its $i$-th column *is of height* $h(i) = k$ if the point $(i - 1, k)$ lies on $P$ and is followed by either an east or diagonal step, $1 \leq i \leq m$. We define the *jump* of the $i$-th column to be the value $j(i) := h(i + 1) - h(i)$, $i = 1, \dots, m - 1$. Lastly, a box $(i, j)$ is called *an outer corner* of $P$ if the point $(i - 1, j - 1)$ lies on $P$ and is followed by a sequence of an east step and a north step.

**Proposition 4.2.** For every $m \in \mathbb{Z}_{>0}$, there exists a bijection between the set of Schröder paths of length $m$ and the set of LLT polynomials of sequences of vertical-strip shapes with $m$ boxes.

*Proof.* We will construct a bijection between Schröder paths of length $m$ and certain LLT graphs, which we will later identify with LLT polynomials of sequences of vertical-strip shapes according to Definition 3.3.

For a path $P$ of length $m$, label box $(i, i)$ by the integer $i$, $1 \leq i \leq m$. Then, construct the LLT graph $G(P)$ with $V(G(P)) = [m]$ and $E_2(G(P)) = \emptyset$ according to the following rules:

1. $(i, j) \in E_1(G(P))$ if and only if $P$ has a diagonal step in the box $(i, j)$;
2. $(i, j) \in E_d(G(P))$ if and only if the box $(i, j)$, $i < j$, lies under $P$.

It is easy to see that the map $P \longmapsto G(P)$ is injective. Indeed, if $P$ and $P'$ agree on the first $k$ steps and differ in the $(k + 1)$-th step which begins at point $(i, j)$, then either

1. $(i, j) \notin E(G(P))$ and $((i, j) \in E_1(G(P'))$ or $(i, j) \in E_d(G(P')))$,
2. $(i, j) \notin E(G(P'))$ and $((i, j) \in E_1(G(P))$ or $(i, j) \in E_d(G(P)))$,
3. $(i, j) \in E_1(G(P))$ and $(i, j) \in E_d(G(P'))$,
4. $(i, j) \in E_d(G(P))$ and $(i, j) \in E_1(G(P'))$.

Observe that $G(P)$ restricted to only edges of type I is a set of disjoint directed paths, and thus corresponds to vertical strips in the sense of Definition 3.3. Furthermore, $E_d(G(P))$ is in a one-to-one correspondence with inversion pairs between vertical strips. Indeed, for two type I directed paths $I = (i_1, \dots, i_s)$ and $J = (j_1, \dots, j_t)$ with $i_1 < j_1$, we must have a directed path $K = (k_1, \dots, k_r)$ of double edges alternating between consecutive vertices of $I$ and $J$ with $k_1 = \max\{i_k \mid k \in [s], i_k < j_1\}$ and either

$$k_r = \begin{cases} \min\{i_k \mid k \in [s], i_k > j_t\} & \text{for } i_s > j_t, \\ \min\{j_k \mid k \in [t], j_k > i_s\} & \text{otherwise.} \end{cases}$$

To obtain the inverse map, it is enough to reverse the procedure. □

### 4.3.1 Correspondence between plane rooted trees and Schröder paths

Let us label the bijection in Proposition 4.2 by $\mu : P \longmapsto \mu(P)$. It turns out that we can connect $\mu$ with the map $\nu$ we introduced in Section 4.2.

**Proposition 4.3.** There exists a bijection $T \leftrightarrow P$ between the set of plane rooted trees with $m$ vertices and the set of Schröder paths of length $m$ which satisfy the following conditions:

1. $P$ is connected (i.e., the only points of $P$ on the main diagonal are $(0, 0)$ and $(m, m)$);
2. the first steps of $P$ are $n$ and $d$; and
3. $P$ has no outer corners,

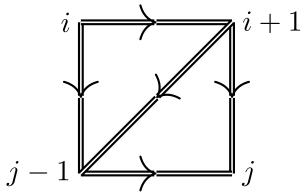
such that $\nu(T) = \mu(P)$.

*Proof.* We claim that the composition $\mu^{-1} \circ \nu$ is the bijection in question. Indeed, $\mu^{-1} \circ \nu$ is an injection from planar rooted trees with $n$ vertices to Schröder paths of length $n$. Therefore, it is enough to prove that the composition maps onto the set of Schröder paths that satisfy the conditions from the statement.

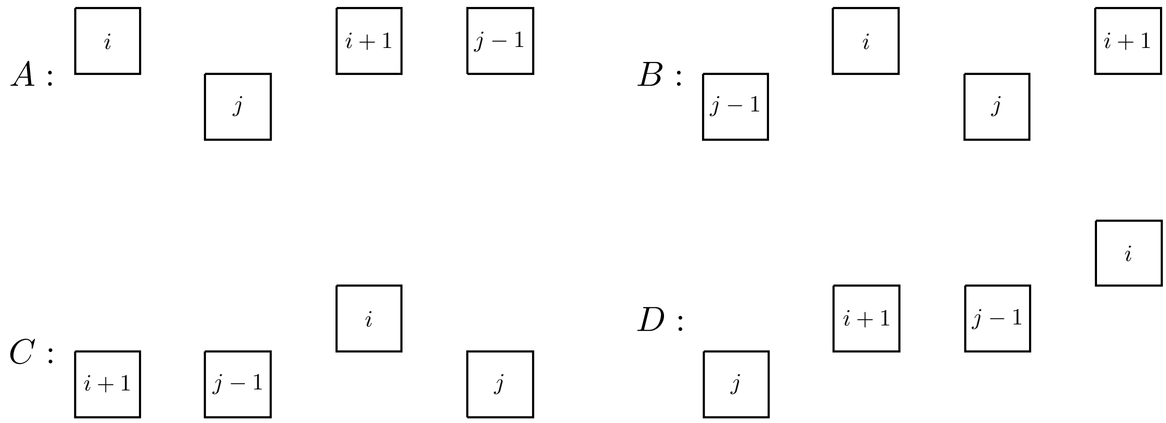
Let $T$ be a plane rooted tree. The bijection given in Definition 4.1 gives a sequence of vertical-strip shapes $\nu(T)$. Let $P$ be the Schröder path that satisfies $\nu(T) = \mu(P)$.

The connectedness of $T$ translates straightforwardly to condition $(1)$ of $P$. Also, observe that the map $T \longmapsto \nu(T)$ forces the cell corresponding to the root $r \in T$ to be the unique cell $\nu(r)$ with the smallest content in $\nu(T)$, and thus, it corresponds to box $(1, 1)$ in $\mu(P)$. Furthermore, $r$'s left-most child maps to the cell below $\nu(r)$ in $(\nu(T))^1$, which translates to $P$ satisfying condition $(2)$ from the statement.

To see that $P$ satisfies condition $(3)$, suppose for contradiction that $P$ has an outer corner in the box $(i, j)$ with no outer corners in boxes $(x, y)$ with $x < i$. This means that in the LLT graph $G = G(P)$ corresponding to $\mu(P)$, we have $(i, i+1), (j-1, j), (i, j-1), (i+1, j-1), (i+1, j) \in E_d(G)$ but $(i, j) \notin E(G)$. Visually, this translates to the subgraph



which further translates to the four possible cell positionings in $\mu(P)$ (here, cells at the same level have the same content in $\mu(P)$):




In particular, there exists no vertex $v \in G(P)$ such that $(i, v) \in E_1(G)$ or $(v, j) \in E_1(G)$.

In cases $A$, $B$, and $C$, the position of $j$ means that in $T$, $j$ is a child of $i$. But by the definition of $\nu$, this contradicts the absence of a vertex $v \in G(P)$ such that $(i, v) \in E_1(G)$. In the case of $D$, the parent of $j$ in $T$ would have content between $c(i)$ and $c(j)$, which would place the vertex between $i$ and $i + 1$, which is impossible.

To see that $\mu^{-1} \circ \nu : T \longmapsto P$ is surjective, it is enough to reverse the reasoning above. I.e., if $P$ is a Schröder path satisfying conditions $(1)$-$(3)$ from the statement, the conditions ensure that $\mu(P) = \nu(T)$ for some tree $T$. $\square$

For an example of the correspondence between Cayley trees and Schröder paths satisfying the relations, see the first two arrows in Fig. 4.1.

### 4.3.2 Correspondence between vertical strips and Dyck paths

As a matter of fact, Proposition 4.3 gives the possibility to express the LLT polynomials corresponding to special sequences of vertical-strip shapes using another class of well-known combinatorial objects.

Recall that a *Dyck path* is a Schröder path with no diagonal steps. Observe that for every Schröder path $P$, we can easily associate with it a Dyck path $D(P)$ by exchanging each diagonal step of $P$ by a sequence of an east step and a north step. Furthermore, if $P$ satisfies conditions from the statement of Proposition 4.3, the first three steps of $D(P)$ are determined: a north step, an east step, and a north step. Thus, we obtain a simple bijection between Schröder paths of length $m$ that satisfy conditions $(1)$-$(3)$ from Proposition 4.3 and Dyck paths of length $m - 1$.

### 4.3.3 Schröder path relations

Similarly to how we introduced local transformations on graphs in Section 3.4.2, we can study relations between Schröder paths. This approach was used by Alexandersson and Sulzgruber [5], who show the following

**Lemma 4.4.** Let $P$ be a Schröder path.

(A) If $P = SneT$ for some paths $S$ and $T$, then

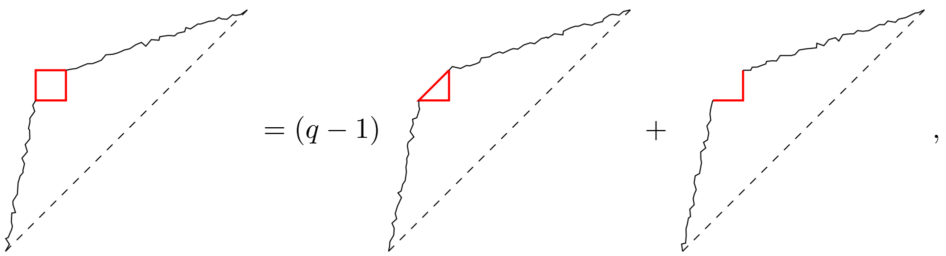
$$\mathrm{LLT}(SneT) = (q-1)\,\mathrm{LLT}(SdT) + \mathrm{LLT}(SenT).$$

(B) If $P = SndReeT$ for some paths $S$, $T$, and $R$ with $S$ ending in $(i,j)$ and $SndR$ ending in $(j,k)$, then

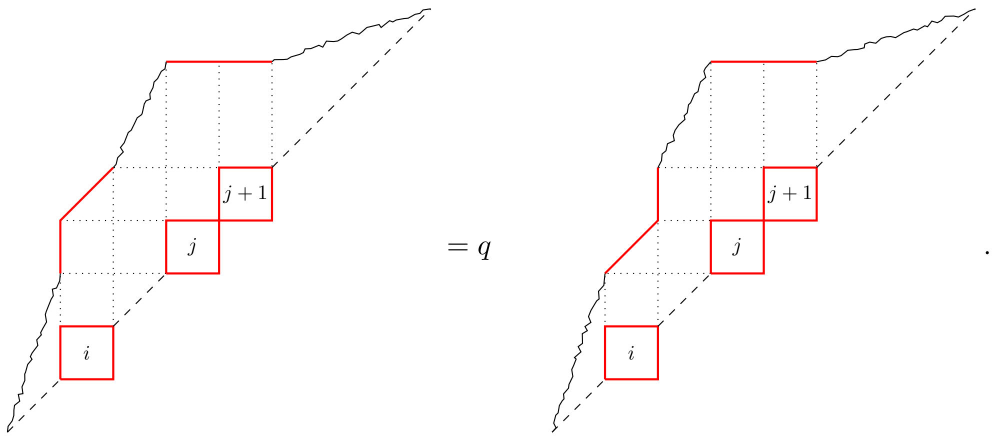
$$\mathrm{LLT}(SndReeT) = q\,\mathrm{LLT}(SdnReeT).$$

The visual representation of the above relations in the spirit of the local transformations presented in Section 3.4.2 takes the form

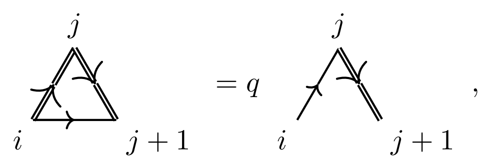
1.




2.




Due to the correspondence between Schröder paths and LLT graphs, these identities translate to identities on the corresponding LLT graphs. To simplify the notation, whenever we state a relation of the form $\sum_{i=1}^{n} f_i(q) \cdot G_i = 0$, where $f_i \in \mathbb{Z}[q]$ and $G_i$ are LLT graphs, we understand it as $\sum_{i=1}^{n} f_i(q) \cdot \mathrm{LLT}(G_i) = 0$ (see Section 3.4.1). What is more, to simplify the formulas even further, we depict pictorially only the subgraph of the bigger graph that involves some changes on the edges. For instance, the second identity locally (i.e., limited to the three vertices associated with the marked boxes) has the following pictorial form:



which formally means that for an LLT graph $G$ associated with the Schröder path $P$ from point $(B)$ of Lemma 4.4, one has

$$\mathrm{LLT}(G) = q\,\mathrm{LLT}(G'),$$

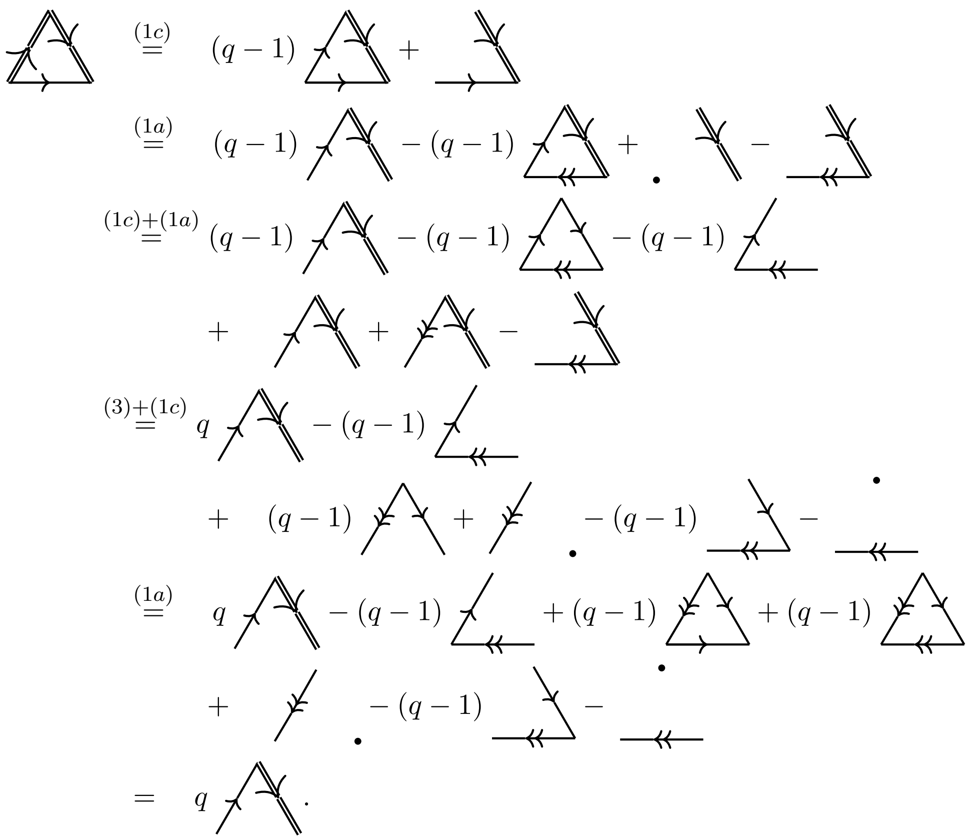
where the only differences between $G$ and $G'$ are on the subgraph induced by the vertex set $\{i, j, j+1\}$ and are described by the picture.

We present a different proof of Lemma 4.4 from the one in [5]. It is an easy manipulation of the relations presented in Lemma 3.4 on graphs corresponding to Schröder paths satisfying the statement of Lemma 4.4 and thus, the relations hold true for those special cases of LLT graphs.

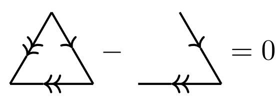
*Proof.* 1. The identity is a straightforward translation of property $(C)$ from Lemma 3.4.

2. From the shape of $P$ we deduce that the only relations that differ between the left- and right-hand sides are between cells $i$, $j$, and $j+1$. In other words, it is enough to look at the objects locally.

   If we use graph relations from Lemma 3.4, we get (for simplicity of notation, we drop the vertex labels)



   The last equality follows from



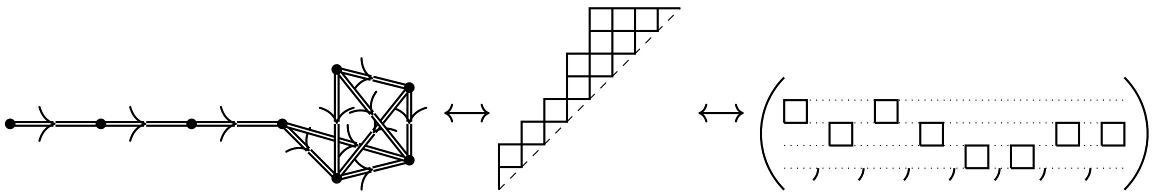


Figure 4.2: The melting lollipop (i.e., its LLT graph variant) $L^{(2)}_{(3,5)}$ and its corresponding Dyck path and sequence of shapes.

(which is a result of Lemma 3.4) and

which results from the following symmetric property of our graph: for every $v > j+1$, the pair $(j, v)$ is an edge of type $A \in \{\rightarrow, \twoheadrightarrow, \Rightarrow\}$ if and only if $(j+1, v)$ is an edge of the same type $A$. Therefore, we can exchange vertices $j$ and $j+1$.

$\square$

## 4.4 Correspondence between melting lollipops and unicellular diagrams

Recall from Definition 3.6 that we obtain the melting lollipop $L^{(k)}_{(l,m)}$ by joining $K_m$ to a path of length $l$ and erasing $k$ edges (see Fig. 4.2 for an example). In fact, there is a straightforward way to translate the notion to the language of LLT polynomials. To be precise, we will denote by $L^{(k)}_{(l,m)}$ the LLT graph on the vertex set $[l+m]$ with $E_1(L^{(k)}_{(l,m)}) = E_2(L^{(k)}_{(l,m)}) = \emptyset$ and $E_d(L^{(k)}_{(l,m)})$ equal to the set $\{(i, i+1) \mid i \in [l]\} \cup (\{(i,j) \mid i,j \in [l+1, l+m], i < j\} \setminus \{(l+1, l+m), \dots (l+1, l+m-k+1)\})$.

In what follows, we only use the LLT variants of melting lollipops and thus, we use for them the same notation as in Definition 3.6. What is more, it is easy to see that melting lollipops correspond to a class of LLT polynomials of unicellular diagrams and those, in turn, correspond to a class of Dyck paths via the map described in Section 4.3.2 (see Fig. 4.2 for an example of the correspondence).

## 4.5 Proof of Theorem 4.1

Before we move on to the proof of Theorem 4.1, let us introduce some notation. Let $\boldsymbol{\lambda}/\boldsymbol{\mu}$ be a sequence of $m$ unicellular shapes and denote the cells by $\square_1, \dots, \square_m$ with $\tilde{c}(\square_1) < \cdots < \tilde{c}(\square_m)$. For a subset $B \subseteq [m]$, write $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(B)$ for the non-directed graph with $V(G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(B)) = B$ and $\{i,j\} \in E(G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(B))$ whenever $0 < |\tilde{c}(\square_i) - \tilde{c}(\square_j)| < m$.

*Proof.* First of all, observe that we can use the Möbius inversion formula for the poset of set partitions to Eq. (3.1) in the LLT case to define the LLT cumulants of melting lollipops recursively by

$$\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu}) = \sum_{\mathcal{B}\in \mathrm{Part}(l+m)} (q-1)^{l+m-|\mathcal{B}|} \prod_{B\in\mathcal{B}} \kappa(B),$$

where $\kappa(B) = \kappa(\lambda^{i_1}/\mu^{i_1}, \ldots, \lambda^{i_r}/\mu^{i_r} \mid B = \{i_1 < \cdots < i_r\})$.

However, as shown in Theorem 3.5, $\kappa(B) = 0$ whenever $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(B)$ is disconnected. Thus, (4.2) is, in fact, equivalent to

$$\begin{aligned}\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu}) &= \sum_{\substack{\mathcal{B}\in \mathrm{Part}(l+m)\\ B\in\mathcal{B}\Rightarrow G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(B) \text{ conncected}}} (q-1)^{l+m-|\mathcal{B}|} \prod_{B\in\mathcal{B}} \kappa(B)\\ &= \sum_{\substack{\mathcal{B}\in \mathrm{Part}(l+m)\\ B\in\mathcal{B}\Rightarrow G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(B) \text{ conncected}}} (q-1)^{l+m-|\mathcal{B}|} \prod_{B\in\mathcal{B}} \sum_{T\subseteq K_B} \mathrm{LLT}(\nu(T)),\end{aligned}$$

where the inner sum runs over all spanning trees of $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(B)$ and we construct $\nu(T)$ according to Definition 4.1.

Lastly, it suffices to interpret the product above as $\mathrm{LLT}(\nu(F))$ with $F$ the forest corresponding to the trees on the components $B \in \mathcal{B}$ to get

$$\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu})(q+1) = \sum_{F} q^{l+m-\#F}\,\mathrm{LLT}(\nu(F))(q+1), \tag{4.4}$$

where the sum runs over all spanning forests of $G_{\boldsymbol{\lambda}/\boldsymbol{\nu}}([l+m])$, $\#F$ denotes the number of connected components of $F$, and $\mathrm{LLT}(\nu(F))$ is the product of $\mathrm{LLT}(\nu(T_i))$ with $T_i$ the connected components of $F$, each rooted in its vertex of the smallest label.

The idea behind the proof is to take the Schröder path corresponding to $\boldsymbol{\lambda}/\boldsymbol{\mu}$ and repeatedly apply Lemma 4.4 to decompose it into a sum of Schröder paths corresponding to spanning forests of $G_{\boldsymbol{\lambda}/\boldsymbol{\nu}}([l+m])$. To be precise, we will show that

$$\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu})(q+1) = \sum_{P} \varphi(P;q)\,\mathrm{LLT}(\mu(P))(q+1),$$

where $\varphi(P;q)$ are polynomials in $q$ dependent on the path $P$ and the sum runs over Schröder paths corresponding to spanning forests of $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}([l+m])$ with each connected component rooted. The coefficients of $\varphi(P;q)$ will correspond to the number of labelings of such forests with integers 1 through $l+m$ which satisfy the conditions from Section 4.2. To be precise, we will describe the coefficients using the shape of $P$ and polynomials

$$W_s(q) \coloneqq \sum_{i=1}^{\tilde{h}(s)-h(s)} \binom{\tilde{h}(s)-h(s)}{i} q^i \in \mathbb{Z}_{>0}[q]$$

where $l+1 \le s \le l+m$ and $\tilde{h}(s)$ and $h(s)$ denote the heights of column $s$ in $L^{(k)}_{(l,m)}$ and $P$, respectively.

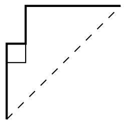
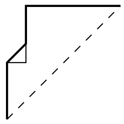
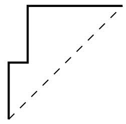
First of all, we apply relation $(A)$ from Lemma 4.4 to each of the first $l$ columns of $L^{(k)}_{(l,m)}$ to reduce each of those columns to minimal height and each diagonal step that the relation gives contributes the factor of $q$ to its corresponding summand. To be precise, we obtain

$$\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu})(q+1) = \sum_{P} q^{d(P)}\,\mathrm{LLT}(\mu(P))(q+1), \tag{4.5}$$

where the sum runs over all Schröder paths $P$ with $h(i) = i$ for $i = 1, \ldots, l$, $h(l+1) = l+m-k$, $h(l+2) = l+m$, and $P$ does not have a diagonal step in the $(l+1)$-th column and $d(P)$ is the number of diagonal steps in $P$.

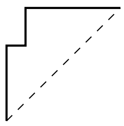
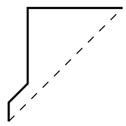
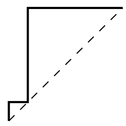
Let $P$ be a Schröder path appearing in (4.5). We will now describe a procedure that utilizes relations $(A)$ and $(B)$ from Lemma 4.4 to decompose the last $m$ columns of $P$. Since it does not affect the first $l$ columns, in the formulas that follow, we draw only the remaining ones.

We begin by reducing the height of the $(l+1)$-th column of $P$. First, we apply relation $(A)$ from Lemma 4.4 to the cell $(l+1, l+m-k)$, which gives

 $= q$  $+$  . (4.6)

Next, to the first summand, we apply relation $(B)$ from Lemma 4.4 to move the diagonal step downwards and repeat until the diagonal step is in the cell $(l+1, l+2)$.

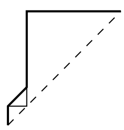
To the second summand from (4.6), we apply $(A)$ from Lemma 4.4 to cell $(l+1, l+m-k-1)$. This will again give us two summands, one with a diagonal step in the first column and one without. To these two, we repeat the reasoning presented until we reduce the first column of the path until no longer possible. Therefore, we end up with

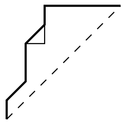
 $= W_{l+1}(q)$  $+$  , (4.7)

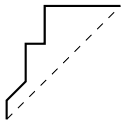
where

$$\begin{aligned} W_{l+1}(q) &= q(q+1)^{\tilde{h}(l+1)-h(l+1)-2} + q(q+1)^{\tilde{h}(l+1)-h(l+1)-3} + \cdots + q \\ &= (q+1)^{\tilde{h}(l+1)-h(l+1)-1} - 1 \\ &= \sum_{i=1}^{\tilde{h}(l+1)-h(l+1)} \binom{\tilde{h}(l+1)-h(l+1)}{i} q^i. \end{aligned}$$

The next steps of the decomposition differ for every summand. For the second shape in (4.7), we reduce the $(l+2)$-th column as we did above. For the first, the exponent of $q$ will determine the jump of the $(l+1)$-th column in the next steps of the decomposition. To be precise, for $1 \le s \le m-k-j(l)$, we decompose the summand with $q^s$ until the $(l+2)$-th column is of height $l+s+1$ (equivalently, until $j(l+1) = s$). In other words, we use the two identities from Lemma 4.4 to get

$$\binom{\tilde{h}(l+1)-h(l+1)}{j(l+1)} q^{j(l+1)}$$  $=$

$$= \binom{\tilde{h}(l+1) - h(l+1)}{j(l+1)} q^{j(l+1)} W_{l+2}(q)$$ 

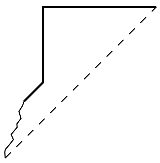
$$+ \binom{\tilde{h}(l+1) - h(l+1)}{j(l+1)} q^{j(l+1)}$$  ,

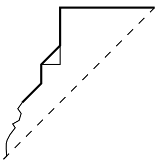
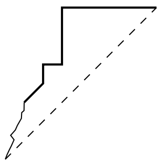
where, analogously to the previous step,

$$W_{l+2}(q) = (q+1)^{\tilde{h}(l+2)-h(l+2)} - 1 = \sum_{i=1}^{\tilde{h}(l+2)-h(l+2)} \binom{\tilde{h}(l+2) - h(l+2)}{i} q^i.$$

In general, assume that we have decomposed the first $2 \leq s \leq m-1$ columns, i.e., that we have managed to express the left-hand side of (4.4) as a sum of Schröder paths whose first $l+s$ columns have been reduced.

Take a summand with coefficient $q^K$ of a path with diagonal steps in columns $l+1 \leq i_1 < \cdots < i_r = s$, $K = j(i_1) + \cdots + j(i_{r-1}) + t$, where $1 \leq t \leq m - h(s)$ (if $i_r < s$ then $K = j(i_1) + \cdots + j(i_r)$ and we reduce the $(s+1)$-th column maximally as in the first step). We lower the $(s+1)$-th column to height $l + K + 1$.

$$\prod_{b=1}^{r-1} \binom{\tilde{h}(i_b) - h(i_b)}{j(i_b)} \cdot \binom{\tilde{h}(i_r) - h(i_r)}{t} q^K$$  $=$

$$= \prod_{b=1}^{r} \binom{\tilde{h}(i_b) - h(i_b)}{j(i_b)} q^K W_{i_r+1}(q)$$  $$+ \prod_{b=1}^{r} \binom{\tilde{h}(i_b) - h(i_b)}{j(i_b)} q^K$$  ,

where $K = j(i_1) + \cdots + j(i_r)$ (meaning that $t = j(i_r)$) and

$$W_{i_r+1}(q) = (q+1)^{\tilde{h}(i_r+1)-h(i_r+1)} - 1 = \sum_{i=1}^{\tilde{h}(i_r+1)-h(i_r+1)} \binom{\tilde{h}(i_r+1) - h(i_r+1)}{i} q.$$

In the end, the above decomposition gives

$$\mathrm{LLT}(\boldsymbol{\lambda}/\boldsymbol{\mu})(q+1) = \sum_P \varphi(P;q)\, \mathrm{LLT}(\mu(P))(q+1), \tag{4.8}$$

where the sum runs over all Schröder paths $P$ contained in $L^{(k)}_{(l,m)}$ (i.e., ones that never go above $L^{(k)}_{(l,m)}$) with $h(l+1) = l+1$ and whose each connected component of length at least 2 (i.e., a section of the path from one point on the diagonal to another without any others in between) begins with $nd$ and has no outer corners, and

$$\varphi(P;q) = q^{l+m-c(P)} \prod_{s=1}^{l} \binom{n - h(i_s)}{n - h(i_s+1)},$$

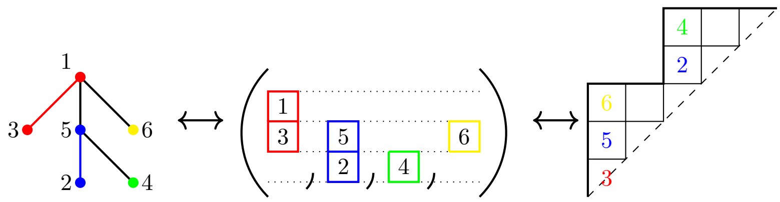


Figure 4.3: A spanning tree with its corresponding sequence of shapes and parking function.

where $c(P)$ is the number of connected components of $P$ and $i_1, \ldots, i_r$ are the columns of $P$ containing diagonal steps.

Therefore, each shape appearing in (4.8) corresponds to a forest in the sense of Section 4.3.1. Furthermore, a path $P$ having diagonal steps in columns $i_1, \ldots, i_r$ corresponds to the vertex $i_s$ having $j(i_s)$ children in the forest.

It remains to show that the number of forests on $[l+m]$ with labelings satisfying the conditions from Section 4.2 which correspond to the same planar drawing $F$ is equal to the multinomial coefficient from $\varphi(P;q)$ with $P$ the Schröder path associated with $F$. In fact, it is a straightforward consequence of the combinatorics of set partitions and thus, we leave the details to the reader. □

## 4.6 Complete graph case

Observe that a special case of a melting lollipop is the complete graph $K_m = L^{(0)}_{(0,m)}$, which corresponds to the full Dyck path $P_m = n^m e^m$ and to the sequence $\boldsymbol{\lambda}/\boldsymbol{\mu} = ((1), \ldots, (1))$ of $m$ unicellular non-skew diagrams.

A straightforward application of Theorem 4.1 to $K_m$ implies the first equality in Corollary 4. Indeed, in this case, we can understand spanning trees as Cayley trees using the correspondence from Section 4.2. What is more, this allows for linking the result to yet another family of well-known combinatorial objects: *parking functions*.

**Definition 4.2.** For $m \in \mathbb{Z}_{>0}$, we say that $f : [m] \to [m]$ is a *parking function on $m$ cars* if $|f^{-1}([i])| \geq i$ for $i = 1, \ldots, m$.

Recall that the maps $\nu$ and $\mu$ give simple ways to translate the vertex labels on a Cayley tree $T$ to cell labels in the sequence $\nu(T)$ and to diagonal box labels in the Schröder path $P$ such that $\nu(T) = \mu(P)$ (see Fig. 4.1).

It is well known that we can represent parking functions geometrically as Dyck paths with special labeling. To be precise, for a Dyck path $D$ of length $m$, we label the boxes to the right of north steps with integers 1 through $m$ so that the numbers increase in columns upwards. The corresponding parking function $f_D : [m] \to [m]$ is then the map, where $f_D(i)$ is equal to the number of the column in which $i$ appears in $D$. As such, in (4.3), we exploit the notation and write $\mu(f)$ for the sequence of unicellular shapes corresponding to the Dyck path associated with $f$.

Also, recall that when we introduced Dyck paths in Section 4.3, we described a simple bijection between a certain subclass of Schröder paths of length $m$ and Dyck paths of length $m-1$. In the complete graph case, we can combine it with Proposition 4.3 to obtain a bijection $T \longleftrightarrow f$ satisfying $\nu(T) = \mu(f)$ with $T$ a Cayley tree on $m$ vertices and $f$ its corresponding parking function on $m-1$ cars (see Fig. 4.3). This assignment explains the second sum in Corollary 4.

**Remark 8.** *Parking functions and Cayley trees are examples of classical combinatorial objects that appear in many different contexts. As such, finding bijections between the two classes that preserve certain statistics is an interesting problem in itself and the above correspondence is an example of that. Indeed, the bijection that naturally appears in this paper in the context of the combinatorics of symmetric functions, is the same as the one found recently in [33, Section 7].*

## 4.7 Directions for further research

In light of (4.1), it would be tempting to say that for all sequences of unicellular shapes, the decomposition conjectured in Eq. (4.1) composes only of trees of the form studied in this chapter, i.e., of LLT polynomials of sequences in which the top cell of the first shape never forms an inversion. Unfortunately, the below example contradicts that.

**Example.** *Let* $\boldsymbol{\lambda}/\boldsymbol{\mu} = ((1),(1),(1,1)/(1),(1,1)/(1))$. *Then*

$$\kappa(\boldsymbol{\lambda}/\boldsymbol{\mu}) = (q^3+3q^2+6q+6)s_{(1^4)} + (q^2+4q+6)s_{(2,1^2)} + (q+3)s_{(2^2)} + s_{(3,1)}.$$

*Furthermore, the following are the unique five sequences of shapes with four cells which appear in the decomposition given by Theorem 4.1:*

$$\begin{aligned}
\mathrm{LLT}((1,1),(1),(1)) &= q^3 s_{(1^4)} + (q^2+q)s_{(2,1^2)} + qs_{(2^2)} + s_{(3,1)},\\
\mathrm{LLT}((1,1,1)/(1),(1,1)) &= qs_{(1^4)} + s_{(2,1^2)} + s_{(2^2)},\\
\mathrm{LLT}((1,1,1),(1,1)/(1)) &= qs_{(1^4)} + s_{(2,1^2)},\\
\mathrm{LLT}((1,1,1),(1)) &= qs_{(1^4)} + s_{(2,1^2)},\\
\mathrm{LLT}((1,1,1,1)) &= s_{(1^4)}.
\end{aligned}$$

*In particular, none of the shapes contains a factor of* $q^2$ *in* $s_{(1^4)}$.

*On the other hand, we have*

$$\mathrm{LLT}((1,1),(1,1,1)/(1)) = q^2 s_{(1^4)} + qs_{(2,1^2)} + q_{(2^2)}.$$

Unfortunately, we were unable to devise even a conjectural formula for the general case of a cumulant of unicellular diagrams but we believe that further studies of the problem may prove fruitful in generalizing Theorem 4.1.

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