APPLICATION OF ALGEBRAIC METHODS IN GEOMETRIC TOMOGRAPHY
ABSTRACT

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Geometric tomography is a branch of mathematics dealing with the retrieval of information about a high-dimensional geometric object from its low-dimensional characteristics [**1**]. Most often these are either its sections by hyperplanes or its projections (shadows) on hyperplanes. The subject naturally overlaps with convex geometry and employs many of its tools. In my research, I follow the idea of associating with every low-dimensional characteristic an algebraic object (e.g. the group of its affine symmetries). Assuming that the hypersurface is sufficiently smooth, this allows us to find certain constraints satisfied by its Taylor polynomial and thus rephrase a geometric assumption in the language of (non)commutative algebra. Such a homological approach usually requires combining many different fields of mathematics, ranging from general topology and abstract algebra to partial differential equations to differential and algebraic geometry.

My dissertation consists of three independent chapters. Each of them illustrates an application of the previously described general paradigm to some particular problem in geometric tomography.

In the first chapter, we will prove that an origin-symmetric star-convex body K with sufficiently smooth boundary and such that every hyperplane section of K passing through the origin is a body of affine revolution, is itself a body of affine revolution. This will give a positive answer to the recent question asked by G. Bor, L. Hernández-Lamoneda, V. Jiménez de Santiago, and L. Montejano-Peimbert, though with slightly different prerequisites. The chapter was already published as an article [**3**].

Let $f \in W^{3,1}_{\mathrm{loc}}(\Omega)$ be a function defined on a connected open subset $\Omega \subseteq \mathbb{R}^2$. In the second chapter, we will show that its graph is contained in a quadratic surface if and only if f is a weak solution to a certain system of 3^{rd} order partial differential equations unless the Hessian determinant of f is non-positive on the whole Ω. Moreover, we will prove that the system is in some sense the simplest possible in a wide class of differential equations, which will lead to the classification of all polynomial partial differential equations satisfied by parametrizations of generic quadratic surfaces. Although we will mainly use the tools of linear and commutative algebra, the theorem itself is also somehow related to holomorphic functions. To perform lengthy computations, we employ a widely used technical computing system Wolfram Mathematica [**2**]. Nevertheless, the proof remains human-surveyable.

An infinitely smooth symmetric convex body $K \subset \mathbb{R}^d$ is called k-separably integrable, $1 \leq k < d$, if its k-dimensional isotropic volume function $V_{K,H}(t) = \mathcal{H}^d(\{\boldsymbol{x} \in K : \mathrm{dist}(\boldsymbol{x}, H^\perp) \leq t\})$ can be written as a finite sum of products in which the dependence on $H \in \mathrm{Gr}(k, \mathbb{R}^d)$ and $t \in \mathbb{R}$ is separated. In the third chapter, we will obtain a complete classification of such bodies. Namely, we will prove that if $d-k$ is even, then K is an ellipsoid, and if $d-k$ is odd, then K is a Euclidean ball. This generalizes the recent classification of polynomially integrable convex bodies in the symmetric case. The chapter is based on a joint work with V. Yaskin.

References

[**1**] R. J. Gardner, *Geometric tomography*, Encyclopedia of Mathematics and its Applications, vol. 13, Cambridge University Press, 2006.
[**2**] Wolfram Research, Inc., *Mathematica, Version 11.0.0*, Champaign, IL, 2016.
[**3**] B. Zawalski, *On star-convex bodies with rotationally invariant sections*, Beiträge zur Algebra und Geometrie (2023).

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